T.A.G., 2016/02/10, Smooth and étale morphisms.
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1. Flat morphisms. [Stacks, TAG 01U2], [H, II.9]

Let $f\colon X \to Y$ be in (RS), the category of ringed spaces, and $x \in X$. Then f is flat at x if $f^\#\colon \mathcal{O}_{Y,fx} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ into a flat $\mathcal{O}_{Y,fx}$-module, i.e., $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,fx}} - \colon \mathcal{O}_{Y,fx}\text{-mod} \to \mathcal{O}_{X,x}\text{-mod}$ is exact.

And: f is flat if it is flat at all $x \in X$.

Examples. 1. For $A \to B$ in (Ring): $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat iff B is flat as A-module.

2. Free modules are flat.

3. If A is a discr. val. ring, and M an A-module, then:
M is flat $\iff$ M is torsion free $\overset{\text{def}}{\iff}$ $M \to \operatorname{Frac}(A) \otimes_A M$ is injective.

4. $\forall A$, $\forall S \subset A$ mult. system: $A \to S^{-1}A$ is flat.

5. $\forall A$, $\forall M$ in A-mod: (M is flat and of finite presentation) $\iff$ (M is locally free and of finite rank.) [Stacks, 00NX]
$\iff$ (M is finitely generated projective).

6. $\mathbb{Z}^{\mathbb{N}}$ is flat as $\mathbb{Z}$-module, but not free...

7. For $A \to B$ in (Rings) and M a flat A-module, $B \otimes_A M$ is flat as B-module.

Exercise. Let k be a field, $k \not\supset \mathbb{F}_2$. Let $\mathbb{Z}/2\mathbb{Z}$ act on $k[x,y]$: 1 acts as id on k, $x \mapsto -x$, $y \mapsto -y$.

(a) Give a presentation of $k[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ (the sub-k-alg. of $\mathbb{Z}/2\mathbb{Z}$-invariants).

(b) Show that $k[x,y]^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow k[x,y]$ is not flat.

(c) Show that $k[x]^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow k[x]$ (same action, $x \mapsto -x$) is flat.

Suggestion: look at it geometrically, look at fibers.

Thm. [Stacks, ...] Let $f\colon X \to S$ in (Sch) be locally of finite presentation, and flat. Then f is open.

2. <u>Smooth morphisms</u>. [H, III.10] only considers schemes of finite type over fields, and that is really insufficient for too many things that we want to do. So we follow [Stacks, 00T1 and 01V4]. 02GZ and 01ZC

<u>Def.</u> Let $f: X \to S$ in (Sch). Then f is <u>smooth</u> iff. $\forall x \in X$ $\exists\, U \subset X$ affine open with $x \in U$, and $\exists\, V \subset S$ affine open with $fx \in V$, $fU \subset V$ and $\exists$ a presentation $\mathcal{O}(V)[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \xrightarrow{\sim} \mathcal{O}(U)$ of $\mathcal{O}(V)$-algebras such that

$$g := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_c} & \cdots & \frac{\partial f_c}{\partial x_c} \end{pmatrix}$$ becomes a unit in $\mathcal{O}(U)$. (Jacobian matrix, Jacobian criterion).

(equivalently: $\mathcal{O}(V)[x_1, \ldots, x_n]/(f_1, \ldots, f_c, g) = 0$, $f_1, \ldots, f_c, g$ have no common zeros.)

Intuitively: locally on X and S, we have c (= codim) equations, whose gradients are lin. indep.
Also: $\mathbb{A}^n_{\mathcal{O}(V)} \xrightarrow{f} \mathbb{A}^c_{\mathcal{O}(V)}$ is a submersion, $U = f^{-1}0$:

$$\begin{array}{ccc} \mathbb{A}^n_{\mathcal{O}(V)} & \longrightarrow & \mathbb{A}^c_{\mathcal{O}(V)} \\ \uparrow & \square & \uparrow 0 \\ U & \longrightarrow & \operatorname{Spec}(\mathcal{O}_V) = V. \end{array}$$

(this should remind you of diff. geometry)
(the local model of diff. geom. applies $/\mathbb{C}$ for $X^{an} \to S^{an}$!)

<u>Remarks:</u> 1. Of course, in algebr. geom., all morphisms are locally given by regular functions, and therefore differentiable. So, smoothness means something else than differentiable. It implies for example that geometric fibres are nonsingular varieties.

2. The number $n-c$ is independent of the presentation and is called the relative dimension at x (it is a loc. const. function $X \to \mathbb{N}$).

3. Smooth $\Rightarrow$ local complete intersection $\Rightarrow$ flat.

4. If f is smooth, then $\Omega^1_{X/S}$ is locally free of rank the relative dimension.

A very nice property of smoothness is that it can be described purely in terms of the "functor of points": $Sch/S \longrightarrow Sets$,
$(T \xrightarrow{g} S) \longmapsto \{h : T \to X \text{ s.t. } f\circ h = g\}$.

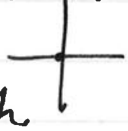
<u>Def.</u> Let $f : X \to S$ in (Sch). Then f is <u>formally smooth</u> if and only if
$\forall\ I \hookrightarrow A \twoheadrightarrow \bar{A}$ with $I^2 = 0$, $\forall$

$$\begin{tikzcd} \operatorname{Spec}(\bar A) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A) \arrow[r, "p"'] \arrow[ur, dashed, "\exists \tilde p"] & S \end{tikzcd}$$

(infinitesimal lifting property).

(Compare with homotopy lifting property from hom. theory.)

<u>Thm.</u> ([Stacks, 02H6]) Let $f : X \to S$ in (Sch). Then TFAE:
(1) f is smooth
(2) f is locally of finite presentation, and formally smooth.
(Rem. See [Stacks, 01ZC] for the functor of points descr. of "loc. of finite pres".)

<u>Example</u> Let k be a field, $S = \operatorname{Spec} k$, $X = \operatorname{Spec}(k[x,y]/(xy))$: 
We show that $X \to S$ is not smooth, bec. not formally smooth.
We take $A = k[\varepsilon]/(\varepsilon^3)$, $\bar A = k[\varepsilon]/(\varepsilon^2)$
Then:

$$\begin{tikzcd} \bar\varepsilon & k[\varepsilon]/(\varepsilon^2) & k[x,y]/(xy) \arrow[l] \arrow[dl, dashed, "\times"] \\ \varepsilon \arrow[u, mapsto] & k[\varepsilon]/(\varepsilon^3) \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

with $\varepsilon \longmapsfrom y$, $\varepsilon \longmapsfrom x$.

<u>Example.</u> Let $k = \bar k$ be an alg. closed field, $f : X \to Y$ a smooth morphism of varieties over k. ~~Please~~ Let $k[\varepsilon] := k[\varepsilon]/(\varepsilon^2)$.
Then

$$\begin{matrix} X(k[\varepsilon]) & = & T_X & \text{is the tangent bundle of } X. \\ \downarrow & & & \\ X(k) & = & X & \end{matrix}$$

Then $T_X \xrightarrow{T_f} f^{-1}T_Y$ is surjective.

<u>Exercise</u> Show that smoothness is preserved under base change and composition. Show that finite inseparable field extensions are not smooth.

3. Étale morphisms. [Stacks, 02GH]

Def. Let $f: X \to S$ be in (Sch). Then f is <u>étale</u> iff f is smooth of relative dimension 0. For $x \in X$: x is <u>étale at x</u> iff $\exists$ open $U \subset X$ with $x \in U$ s.t. $f|_U : U \to S$ is étale.

Thm Let $f: X \to S$ in (Sch). Then TFAE:

1) f is étale,
2) f is loc. of finite presentation and <u>formally étale</u>: $\forall\, I \hookrightarrow A \twoheadrightarrow \bar{A}$ with $I^2 = 0$ $\forall$
$$\begin{tikzcd} \mathrm{Spec}(\bar{A}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "p"'] \arrow[ur, dashed, "\exists!\tilde{p}"] & S, \end{tikzcd}$$
3) $\forall x \in X$ $\exists\, U \subset X$ affine open with $x \in U$, $\exists\, V \subset S$ affine open with $f(U) \subset V$, and a presentation of $\mathcal{O}(V)$-algebras $\mathcal{O}(V)[x]_g/(f) \xrightarrow{\sim} \mathcal{O}(U)$ s.t. f is monic and f' a unit in $\mathcal{O}(U)$.
4) f is flat, locally of finite presentation, and the geometric fibres of f are disjoint $\&$ unions (discrete topology!) of copies of the base. (without the flatness condition, this is called "G-unramified" in [Stacks, 02G3])

Examples. 1). Let k be a field, X a k-scheme. Then X is étale $/k$ iff X is discrete, $\forall x \in X$ $\underset{\mathcal{O}_{X,x}}{\overset{\kappa(x)}{\shortparallel}}$ is a finite separable field extension of k.

2) Let A be a ring, $f \in A[x]$ monic, $B := A[x]/(f)$.
Then $A \to B$ is étale iff $\mathrm{discr}(f) \in A^\times$.

3) Let X be finite etale over $\mathbb{Z}$. Then $X \simeq \coprod_{\text{finite}} \mathrm{Spec}\,\mathbb{Z}$, $\mathcal{O}(X) \cong \mathbb{Z}^n$ for some n.

<u>Exercise</u>. Let $\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$ in (Sch). Then h & g étale $\Rightarrow$ f étale.

<u>Exercise</u>. Let $f: X \to S$ be étale. Then $X \xrightarrow{\text{diag}} X \times_S X$ is an open immersion. Make this explicit for $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A[x]/(g))$, g monic, $\mathrm{discr}(g) \in A^\times$.

<u>Exercise</u>. The notion of étale morphisms is essential to have a kind of "implicit function theorem" in algebraic geometry. Think about this, and make a precise statement.