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Some examples of sheaves on etale sites.

1. The small étale site of a point.

Let k be a field, $S := \mathrm{Spec}(k) = \{s\}$, $k \to k^{\mathrm{sep}}$ a separable closure, $\bar{s} = \mathrm{Spec}(k^{\mathrm{sep}}) \longrightarrow S$ the corresponding geometric point, $\Gamma := \mathrm{Aut}_k(k^{\mathrm{sep}})$ with its profinite topology.

We recall (Pol's talk): $S_{\acute{e}t}$ objects $\begin{matrix} X \\ f\downarrow \text{étale} \\ S \end{matrix}$ morphisms
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd},$$
(full subcat. of Sch/S),
coverings: jointly surjective families
$$\begin{tikzcd} \{X_i \arrow[r] \arrow[dr] & X\}_{i\in I} \arrow[d] \\ & S \end{tikzcd}$$

From Raoul: $S_{\acute{e}t} \longrightarrow \Gamma\text{-Sets} :=$ cat. of discrete continuous Γ-sets

equivalence

$X/S \longmapsto X(\bar{s})$

$(\mathrm{Spec}(A) \longmapsto \mathrm{Hom}_{k\text{-alg}}(A, k^{\mathrm{sep}}))$

(See SP 03QR.)

$\mathrm{Spec}\left(\mathrm{Hom}_{\Gamma\text{-set}}(M, k^{\mathrm{sep}})\right) \longleftarrow M$ for M finite

$\coprod_{i\in I} \mathrm{Spec}\left(\mathrm{Hom}_{\Gamma\text{-set}}(M_i, k^{\mathrm{sep}})\right) \longleftarrow M = \coprod_{i\in I} M_i$ M_i finite

Now sheaves: $\mathrm{Sh}(S_{\acute{e}t}) \longrightarrow \Gamma\text{-sets}$ (continuous action for discr. top)

$$\mathcal{F} \longmapsto \mathcal{F}_{\bar{s}} := \operatorname*{colim}_{\substack{k \subseteq K \subseteq k^{\mathrm{sep}} \\ \text{finite}}} \mathcal{F}(\mathrm{Spec}\, K) = \operatorname*{colim}_{\substack{U \subset \Gamma \text{ open} \\ \text{subgroup}}} \overset{(\mathcal{F}_{\bar{s}})^U}{\overset{\shortparallel}{\overline{\mathcal{F}(k^{\mathrm{sep},U})}}}$$

is an equivalence,
see SP 03QT.

In particular: all sheaves are representable by étale k-schemes.

Example. 1. $\mathbb{G}_m$: $(X \to \mathrm{Spec}\, k) \mapsto \mathcal{O}(X)^\times$, $\mathbb{G}_{m,\bar{s}} = k^{\mathrm{sep},\times} + \Gamma$-action.
(exercise: what is the étale k-scheme representing this sheaf?)

2. Constant sheaves. Let M be a set. This gives the sheaf
$M_k : X \mapsto \{\mathrm{top}(X) \xrightarrow{f} M \text{ loc. const.}\}$.
exercise: what is $(M_k)_{\bar{s}}$, what is the étale k-scheme representing it?
SP03YZ

Why is the fibre functor an equivalence?

$S_{ét}$ Γ-Sets

$X = \coprod$ conn. comp'ts $M = \coprod$ orbits, non-empty transitive Γ-sets

$$\begin{tikzcd} k \arrow[r] \arrow[d] & K \arrow[dl, dashed, hook] \\ k^{sep} & \end{tikzcd}$$

fin. separable field ext'n

$$\begin{tikzcd} m & M \\ e \arrow[u, mapsto] & \Gamma/\Gamma_m \arrow[u, "\wr"] \end{tikzcd}$$

M transitive, non-empty

$k \subset K \subset k^{sep} \supset (k^{sep})^{\Gamma_m}$ now use classical Galois correspondence.

Also: $\forall$ $$\begin{tikzcd}[column sep=tiny, row sep=tiny] & K_1 \arrow[dr, phantom, "\subset"] & \\ k \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & k^{sep} \\ & K_2 \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$, $\forall$ $$\begin{tikzcd} K_1 \arrow[rr, "f"] & & K_2 \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$ $\exists$ $\sigma \in \Gamma$ inducing f.

Important exercise. Let
$$\begin{tikzcd} K \arrow[rr] & & L \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$
fields, finite sep. /k, K → L Galois, $G := \mathrm{Gal}(L/K)$.

Then $\mathrm{Spec}(K) \leftarrow \mathrm{Spec}(L)$ is an étale cover.

$$\begin{tikzcd} \mathrm{Spec}(K) & \mathrm{Spec}(L) \arrow[l] & \mathrm{Spec}(L \otimes_K L) \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_2"] \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r] & L \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & L \otimes_K L \end{tikzcd}$$

$$x \xmapsto{p_1^*} x \otimes 1$$

$$x \xmapsto[p_2^*]{} 1 \otimes x$$

Now $L \otimes_K L \longrightarrow \prod_{g \in G} L$ is an isomorphism of K-algebras, even of L-algebras

$$x \otimes y \longmapsto (g \mapsto x \cdot g(y))$$

Use this to show that for $\mathcal{F}$ a presheaf on Set,

$$\begin{tikzcd} \mathcal{F}(\mathrm{Spec}\, L) \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & \mathcal{F}(\mathrm{Spec}(L \otimes_K L)) \end{tikzcd}$$ has

equaliser $\mathcal{F}(\mathrm{Spec}\, L)^G$.

2. $\mathbb{G}_m$ is a sheaf. Kummer sequences. (SP 03PK)

Let S be a scheme.

As Pol said, $\forall\ X \to S$ in Sch/S, $(T \to S) \mapsto X(T) := \left\{ \begin{tikzcd} & X \arrow[d] \\ T \arrow[ur, dashed] \arrow[r] & S \end{tikzcd} \right\}$

is a sheaf for the etale topology (and fppf). Maybe we will look at the proof later.

Example: $\mathbb{G}_{m,S}$:

$$\begin{tikzcd} \mathbb{G}_{m,S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[x,y]/(xy-1)) \arrow[d] \\ S \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

Then $\forall\ T \to S$: $\mathbb{G}_m(T) = \operatorname{Hom}_{Sch}(T, \mathbb{G}_m) = \mathcal{O}(T)^\times$.

So we have a sheaf of groups.

Let $n \in \mathbb{Z}_{\geq 1}$. Then $\mu_{n,S} : (T \to S) \mapsto \{ z \in \mathcal{O}(T)^\times : z^n = 1 \}$ is represented by the pullback to S of the $\mathbb{Z}$-scheme $\operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1))$.

So we have the sequence: $0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \to \mathbb{G}_{m,S} \to 0$

s.t. $\forall\ T \to S$: $0 \to \mu_n(T) \hookrightarrow \mathcal{O}(T)^\times \longrightarrow \mathcal{O}(T)^\times \to 0$

$$f \longmapsto f^n$$

Lemma 1. The sequence is left-exact (for any topology, actually, as presheaves)

2. If $n \in \mathcal{O}(S)^\times$, i.e., S is a $\mathbb{Z}[1/n]$-scheme, then it is also exact on the right for the etale topology

3. The sequence is exact on the right for the fppf topology.

Proof. 1 is clear. For 2 and 3, let $\overset{\operatorname{Spec} A}{\overset{\shortparallel}{T}} \to S$ be an S-scheme, and let $a \in \mathbb{G}_{m,S}(T) = \mathcal{O}(T)^\times = A^\times$.

Then

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "(\cdot)^n"] \arrow[dr, phantom, "\square"] & \mathbb{G}_m \\ \operatorname{Spec}\left(\frac{A[x]}{(x^n - a)}\right) \arrow[u] \arrow[r] & T = \operatorname{Spec} A \arrow[u, "a"'] \end{tikzcd}$$

is fppf, even finite loc. free rank n, and étale if n is inv. in A.