Bas Edixhoven, TAG 2016/04/13; 2 × 45 minutes.

1. $C^{(n)}$ represents Div^n_C. (<u>Ref</u>: deag596.~~html~~, §9.)
<u>Situation</u>: k a field, C/k smooth of rel.-dim. 1, quasiprojective.

Then $C^{(n)} := S_n \backslash C^n$ as in Andrea's talk.
<u>Def</u>. Let $T \in \mathrm{Sch}/k$. An effective relative Cartier divisor on C_T of degree n is a closed subscheme D of C_T that is finite loc. free of rank n over T whose sheaf of ideals I_D is loc. generated by 1 element that is not a zero-divisor.
<u>Rem</u> For $P \in C(T)$, $P(T) \subset C(T)$ is such a thing, of degree 1.
Can add: $I(D_1+D_2) = I(D_1)\cdot I(D_2)$, $\deg(D_1+D_2) = \deg D_1 + \deg D_2$.
Can subtract: $D_1 + D_2 = D_1 + D_3 \Rightarrow D_2 = D_3$. Is functorial in T:
<u>Def</u> For $n \geq 0$, $\mathrm{Div}^n_C : \mathrm{Sch}/k \longrightarrow \mathrm{Set}$, $T \mapsto \{\text{eff. rel. } C\text{-div. on } C_T \text{ of deg. } n\}$.

We have:
$$\begin{tikzcd} C^n \arrow[r, "p"] \arrow[d, "\pi"'] & \mathrm{Div}^n_C \\ C^{(n)} \arrow[ur, "\bar{p}"] & \end{tikzcd} \qquad \underset{\in C^n(T)}{(P_1, \ldots, P_n)} \mapsto P_1(T) + \cdots + P_n(T)$$
As Div^n_C not yet known to be a scheme, make $\bar{p}$ "directly"!

<u>Thm</u>. $\bar{p}$ is an isomorphism (hence: Div^n_C is repr., we know $C^{(n)}$ as functor).
<u>Rem</u>. This is not so easy to prove at points where the divisor is not reduced.
For $C = \mathbb{P}^1_k$ it is easy, and instructive (see Andrea's notes).
The following weaker version is easier.
Let $\Delta \subset C^n$ be the closed subset $\bigcup_{i<j} \mathrm{pr}_{i,j}^{-1}\Delta$, where $\Delta \subset C \times_k C$ diag.

Let $\mathrm{Div}^{n,et}_C \subset \mathrm{Div}^n_C : T \mapsto \{D \in \mathrm{Div}^n_C(T) : \forall \text{ geom. pt. } \mathrm{Spec}(K) \to T,\ D_K \text{ reduced}\}$.
$= \{D \in \mathrm{Div}^n_C(T) : D \to T \text{ is étale}\}$.

Then:
$$\begin{tikzcd} C^n - \Delta \arrow[r, "p"] \arrow[d, "\pi"'] & \mathrm{Div}^{n,et}_C \\ C^{(n)} - \Delta \arrow[ur, "\bar{p}", "\sim"'] & \end{tikzcd}$$
.

<u>Sketch of proof</u>: produce the inverse of $\bar{p}$. For $T \to \mathrm{Spec}(k)$ and $D \in \mathrm{Div}^{n,et}_C(T)$, $\underset{\text{étale}}{D \to T}$ loc. isom. to $\coprod_{1 \leq i \leq n} T$.

2. $\mathrm{Pic}_{C/k}$ is representable if $C(k) \neq \emptyset$. Ref. deagrg6.html, §10

Situation: add: C/k projective, geom. irreducible, given $P \in C(k)$.

Recall from Nikitas's talk:

Def. $\mathrm{Pic}_{C/k} : \mathrm{Sch}/k \to \mathrm{Ab}$, $T \mapsto \mathrm{Pic}(C_T)/\mathrm{Pic}(T) =$
$= \{ \text{isom. classes of } (\mathcal{L}, r) : \mathcal{L} \text{ inv. } \mathcal{O}_{C_T}\text{-module}, \ r : \mathcal{O}_T \xrightarrow{\sim} P_T^* \mathcal{L} \}$

Rem. For T/k, $\mathcal{L}$ inv. $\mathcal{O}_{C_T}$-module, $T \xrightarrow{\chi(\mathcal{L})} \mathbb{Z}$, $t \mapsto \chi(\mathcal{L}|_{C_t})$ is loc. constant (flatness). Hence we can define
$\deg \mathcal{L} : T \to \mathbb{Z}$, $t \mapsto \chi(\mathcal{L}) + g - 1$, where $g = $ genus of $C = \dim_k H^1(\mathcal{O}_C)$.
Then $T = \coprod_{n \in \mathbb{Z}} \deg(\mathcal{L})^{-1}\{n\}$. Hence $\mathrm{Pic}_{C/k} = \coprod_{n \in \mathbb{Z}} \mathrm{Pic}^n_{C/k}$.

Equivalently:
$$\begin{tikzcd} \mathrm{Pic}^0_{C/k} \arrow[r, hook] & \mathrm{Pic}_{C/k} \arrow[r, two heads, "\deg"] & (\mathbb{Z})_k \arrow[l, bend left] \end{tikzcd}.$$
$$P \longleftarrow\!\shortmid 1$$

Thm. a. $\mathrm{Pic}_{C/k}$ is representable (hence every $\mathrm{Pic}^n_{C/k}$ is, all $\cong \mathrm{Pic}^0_{C/k}$)
b. $\mathrm{Pic}^0_{C/k}$ is projective, $J := \mathrm{Pic}^0_{C/k}$ (jacobian variety).
c. $C \xrightarrow{i} J$, $Q \mapsto \mathcal{O}(Q-P)$ induces
$$\begin{tikzcd} \mathrm{Pic}_{J/k} \arrow[r, "i^*"] & \mathrm{Pic}_{C/k} \\ \text{conn. comp. of } 0 = \mathrm{Pic}^0_{J/k} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\sim"] & J \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$
the natural principal polarisation of J.

Main idea of proof of a.
$$\begin{tikzcd} C^{(g)} \arrow[d, phantom, "\shortparallel"] & \\ \mathrm{Div}^g_C \arrow[r, "\varphi"] & \mathrm{Pic}^g_{C/k} \\ C^{(g)} - \Delta = \mathrm{Div}^{g,\mathrm{et}}_C \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd}$$
$D \mapsto [\mathcal{O}_{C_T}(D)]$ is birational

Hand waiving style: for $\mathcal{L}$ in $\mathrm{Pic}^g(C_t)$, $\varphi^{-1}[\mathcal{L}] = \dfrac{H^0(C_t, \mathcal{L}) - \{0\}}{t^*}$,
so φ induces an open immersion on the open set U of $C^{(g)}$ where $h^0(C_t, \mathcal{L}) = 1$.

More seriously: $\forall\, T/k$, $\forall\, \mathcal{L}$ in $\mathrm{Pic}^g(C_T)$:
$$\begin{tikzcd} C_T \arrow[d, "f"'] & \mathcal{L} \\ T & \end{tikzcd}$$
$(R^1 f_*)(\mathcal{L})$ is coherent,
let $U := T - \mathrm{supp}\,(R^1 f_*)\mathcal{L}$, then $f_* \mathcal{L}|_{C_U}$ is loc. free rank 1, its formation commutes with base change, $\exists!$ D in $\mathrm{Div}^g_C(U)$ s.t. $\mathcal{L}|_{C_U} \cong \mathcal{O}_{C_U}(D)$ mod $\mathrm{Pic}(U)$.

Applying this to $C^{(g)}$ this gives a dense open $U \subset C^{(g)}$ on wich φ induces an <u>open immersion</u>:

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \mathrm{Pic}^g_{C/k} \\ \uparrow & \square & \uparrow \\ V & \hookrightarrow & T \end{array}$$

for all $T \to \mathrm{Pic}^g_{C/k}$ the pull back of φ exits (V is a scheme) and is op. imm.

<u>Now assume $k = \bar{k}$.</u> Then the translates $t_x \circ \varphi$, $x \in \mathrm{Pic}^0(C)$, cover $\mathrm{Pic}^g_{C/k}$, and $\mathrm{Pic}^g_{C/k}$ is represented by the scheme obtained by glueing these charts. See §10.4.

$C^{(g)} \longrightarrow \mathrm{Pic}^g_{C/k}$ is surjective, hence $\mathrm{Pic}^g_{C/k}$ is proper.

Hence $\mathrm{Pic}^0_{C/k}$ is proper, hence an ab. var., hence projective. (/field!)

<u>Rem.</u> Θ-divisor in $\mathrm{Pic}^{g-1}_{C/k}$: "$\{\mathcal{L} \in \mathrm{Pic}^{g-1}(C) : h^0(\mathcal{L}) \geq 1\}$". Ample!

<u>But what if $k \neq \bar{k}$?</u>

Then take $k \to k'$ finite Galois, gr. Γ, s.t. $\mathrm{Pic}^0_{C_{k'}/k'}$ repr., say by J' over k'.

Then $\forall \gamma \in \Gamma$: $J' \xrightarrow[\sim]{\beta(\gamma)} J'$ $\quad$ Γ acts (on the right) on

$$\begin{array}{ccc} C_{k'} & \xrightarrow[\sim]{\alpha(\gamma)} & C_{k'} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \xrightarrow[\mathrm{Spec}(\gamma)]{\sim} & \mathrm{Spec}(k') \end{array} \qquad \begin{array}{c} J' \\ \downarrow \\ \mathrm{Spec}(k') \end{array} \qquad \begin{array}{c} J := J'/\Gamma \\ \downarrow \\ \mathrm{Spec}(k). \end{array}$$

exists bec. J' projective.

This is called Galois descent of schemes.

Question: what does J represent?

Answer: if $C(k) \neq \emptyset$, then $\mathrm{Pic}^0_{C/k} : T \longmapsto \mathrm{Pic}(C_T)/\mathrm{Pic}(T)$.

<u>Example</u> $k := \mathbb{R}$, $C = V(x^2+y^2+z^2) \subset \mathbb{P}^2_{\mathbb{R}}$. Note $C(\mathbb{R}) = \emptyset$; $C_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$.

Then $\mathrm{Pic}_{C/k}$ is not a sheaf for the étale topology:

$$\mathrm{Pic}_{C/\mathbb{R}}(\mathbb{C}) = \mathrm{Pic}(C_{\mathbb{C}}) \xrightarrow[\sim]{\deg} \mathbb{Z}, \text{ with trivial } \mathrm{Gal}(\mathbb{C}/\mathbb{R})\text{-action.}$$

$$\mathrm{Pic}_{C/\mathbb{R}}(\mathbb{R}) = \mathrm{Pic}(C) \xrightarrow[\sim]{\deg} 2\mathbb{Z}.$$

So, in general, one should define $\mathrm{Pic}_{C/k}$ as the étale sheafification of $T \longmapsto \mathrm{Pic}(C_T)/\mathrm{Pic}(T)$: $f: C \to \mathrm{Spec}\, k$, $(R^1 f_*)\mathbb{G}_m$.

$$f_*: \underset{\mathrm{Ab}}{\wedge}(\mathrm{Sch}/C)_{et} \to \underset{\mathrm{Ab}}{\wedge}(\mathrm{Sch}/k)_{et}$$

3. fpqc-descent of modules. Ref. {MSc thesis Zomervrucht, Vistoli's notes descent.pdf, § 4.2

We first consider the situation in Set. $(X \to S) \mapsto (X_T \to T)$

Let $T \xrightarrow{f} S$ be surjective (in Set). Gives $f^{-1}: \mathrm{Set}/S \to \mathrm{Set}/T$.

Then, for $t_1, t_2 \in T$ with $f(t_1) = f(t_2)$, $(X_T)_{t_1} = X_{f(t_1)} = X_{f(t_2)} = (X_T)_{t_2}$.

Moreover, for $s \in S$, $t_1, t_2, t_3 \in f^{-1}\{s\}$, $(X_T)_{t_1} = (X_T)_{t_2} = (X_T)_{t_3}$ compatible

Now given $Y \to T$ with (given) $\forall s \in S, \forall t_1, t_2 \in f^{-1}\{s\}: Y_{t_1} \xrightarrow[\sim]{\varphi_{t_1,t_2}} Y_{t_2}$

such that $\forall s \in S, \forall t_1, t_2, t_3 \in f^{-1}\{s\}: \varphi_{t_1,t_3} = \varphi_{t_2,t_3} \circ \varphi_{t_1,t_2}$,

we get an $X \to S$ +
$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$
, and these determine $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ up to unique isomorphism.

(To make such an X/S, choose a point in every fibre of f, i.e., a section of f, or take a suitable limit.)

Reformulation without using elements of sets with the same image in $S(U)$ (resp. $S(V)$):

$\forall U, \forall t_1, t_2 : U \to T$, we want $\varphi_{t_1,t_2} : t_1^* Y \xrightarrow{\sim} t_2^* Y$ s.t.

$\forall V, \forall t_1, t_2, t_3 : V \to T: \varphi_{t_1,t_3} = \varphi_{t_2,t_3} \circ \varphi_{t_1,t_2}$.

(Rem. universal case: $U = T \times_S T$, $t_i = pr_i$; $V = T \times_S T \times_S T$, $t_i = pr_i$.)

Now let $T \xrightarrow{f} S$ be an fpqc cover in Sch, $Y \to T$.

A descent datum for $Y \to T$ rel. to $T \xrightarrow{f} S$ is:

an isom. $\varphi: pr_1^{-1} Y \xrightarrow{\sim} pr_2^{-1} Y$ on $T \times_S T$, such that

on $T \times_S T \times_S T$: $(p_1, p_3)^{-1} \varphi = (p_2, p_3)^{-1} \varphi \circ (p_1, p_2)^{-1} \varphi$.

Then we have $\mathrm{Sch}/S \xrightarrow{f^{-1}} \mathrm{Sch}/T$ + desc.dat., fully faithful (by Pol's lecture). Unfortunately not always essentially surjective.

But f^{-1} is ess. surjective when restricted to schemes that are affine $/S$, resp. $/T$. To prove this, one treats the analogous case of quasi-coh. $\mathcal{O}$-modules. The reduction to S and T affine is easy.

So now $A := \mathcal{O}(S)$, $B := \mathcal{O}(T)$, $A \xrightarrow{f^*} B$ faithfully flat.

Let M be a B-module and φ a descent datum on M to A:

$\varphi: pr_1^* M \xrightarrow{\sim} pr_2^* M$ isom. of $B \otimes_A B$-modules, $pr_1^*: B \to B \otimes_A B$, $b \mapsto b \otimes 1$

$p_2^* : B \longrightarrow B \otimes_A B$, $b \mapsto 1 \otimes b$. Then $(p_1,p_3)^*\varphi = (p_2,p_3)^*\varphi \circ (p_1,p_2)^*\varphi$, isomorphisms of $B^{\otimes 3}$-modules ($\otimes$ over A).

the $B^{\otimes 2}$-module

<u>Thm</u> (Grothendieck, I presume). Let $N = \{ m \in M : \varphi(p_1^* m) = p_2^* m \text{ in } p_2^* M \}$.
Then N is a sub-A-module of M.
The B-module morphism $B \otimes_A N \longrightarrow M$, $b \otimes n \mapsto b \cdot n$, is an isomorphism, compatible with descent data: pull-back on LHS, φ on RHS.

<u>Proof</u>. (I follow Zomervrucht, but written using projection maps, not tensors...).
We have the equaliser diagram $N \xrightarrow{i} M \underset{\varphi^{-1} \circ p_2^*}{\overset{p_1^*}{\rightrightarrows}} p_1^* M$

This diagram lies over

$$\begin{tikzcd} S & T \arrow[l, "f"'] & T^2 \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_2"] \\ T \arrow[u, "f"] & T^2 \arrow[l, "p_2"] \arrow[u, "p_1"] & T^3 \arrow[l, shift right, "(p_1,p_3)"'] \arrow[l, shift left, "(p_2,p_3)"] \arrow[u, "(p_1,p_2)"'] \end{tikzcd}$$

We pull it back to

giving us the diagram $f^* N \xrightarrow{f^*(i)} p_1^* M \underset{((p_1,p_2)^*\varphi^{-1}) \circ (p_2,p_3)^*}{\overset{(p_1,p_3)^*}{\rightrightarrows}} p_1^* M$.

This last diagram is isomorphic to the equaliser diagram from Pol's lecture obtained from viewing M as an A-module:

$$\begin{tikzcd} f^* N \arrow[r, "f^*(i)"] \arrow[d] & p_1^* M \arrow[r, shift left, "(p_1,p_3)^*"] \arrow[r, shift right, "((p_1,p_2)^*\varphi^{-1}) \circ (p_2,p_3)^*"'] \arrow[d, "\varphi"', "\wr"] & p_1^* M \arrow[d, "\wr"', "(p_1,p_3)^*\varphi"] \\ M \arrow[r, "p_2^*"'] & p_2^* M \arrow[r, shift left, "(p_1,p_3)^*"] \arrow[r, shift right, "(p_2,p_3)^*"'] & p_3^* M \end{tikzcd}$$

Both rows are equalisers, hence $f^* N \longrightarrow M$ is an isomorphism.

For commutativity of the lower RH square:

$$\begin{tikzcd} p_1^* M \arrow[r, "(p_1,p_2)^*\varphi"] \arrow[dr, "(p_1,p_3)^*\varphi"'] & p_2^* M \arrow[d, "(p_2,p_3)^*\varphi"] \arrow[dl, phantom, "\circledcirc"] & p_1^* M \text{ on } T^2 \arrow[l, "(p_2,p_3)^*"'] \arrow[d, "\varphi"] \\ & p_3^* M & p_2^* M \text{ on } T^2 \arrow[l, "(p_2,p_3)^*\varphi"] \end{tikzcd}$$

Thm. Let X be a scheme. Then the natural morphisms of groups

$$H^1(X_{Zar}, \mathbb{G}_m) \to H^1(X_{et}, \mathbb{G}_m) \to H^1((Sch/X)_{et}, \mathbb{G}_m) \to H^1((Sch/X)_{fpqc}, \mathbb{G}_m)$$

are isomorphisms.

Proof. In all cases, $\mathbb{G}_m = \underline{Aut}(\mathcal{O})$ (that $\mathbb{G}_m$ and $\mathcal{O}$ are sheaves was explained in Pol's lecture). By Luca's lecture: these H^1's are the sets of isomorphism classes of twists of $\mathcal{O}$: the $\mathcal{O}$-modules $\mathcal{L}$ that are locally (for the corresponding topology) isomorphic to $\mathcal{O}$. We claim that such $\mathcal{L}$ are all already Zariski-locally isomorphic to $\mathcal{O}$.

Let $\mathcal{L}$ be an $\mathcal{O}$-module on $(Sch/X)_{fpqc}$ that is locally isom. to $\mathcal{O}$. We want to show that $\mathcal{L}|_{X_{Zar}}$ is locally isom to $\mathcal{O}$.
We may (and do) assume that X is affine, $X = \operatorname{Spec} A$. Then there is $A \to B$ faithfully flat such that $f^*\mathcal{L}$ is isomorphic to $\mathcal{O}$. Pick an $X \xleftarrow{f} Y$ isomorphism $\alpha : \mathcal{O} \xrightarrow{\sim} f^*\mathcal{L}$. The natural descent datum of $\mathcal{L}$ rel. to $Y \to X$ gives a descent datum on $\mathcal{O}$ rel. to $Y \to X$. Hence, by descent of quasicoherent $\mathcal{O}$-modules, $\mathcal{L}|_{X_{Zar}}$ is quasi-coherent, and loc. free of rank 1. $\boxtimes$