

Bas Edixhoven, TAG 2016/04/13; 2 x 45 minutes.

1.

1. $C^{(n)}$ represents Div_C^n . (Ref: dea9596.html, §9.)

Situation: k a field, C/k smooth of rel-dim. 1, quasi-projective.

Then $C^{(n)} := S_n \backslash C^n$ as in Andrea's talk.

Def. Let $T \in \text{Sch}/k$. An effective relative Cartier divisor on C_T of degree n is a closed subscheme D of C_T that is finite loc. free of rank n over T whose sheaf of ideals I_D is loc. generated by 1 element that is not a zero-divisor.

Rem. For $P \in C(T)$, $P(T) \subset C(T)$ is such a thing, of degree 1.

Can add: $I(D_1 + D_2) = I(D_1) \cdot I(D_2)$, $\deg(D_1 + D_2) = \deg D_1 + \deg D_2$.

Can subtract: $D_1 + D_2 = D_1 + D_3 \Rightarrow D_2 = D_3$. Is functorial in T :

Def. For $n \geq 0$, $\text{Div}_C^n : \text{Sch}/k \rightarrow \text{Set}$, $T \mapsto \{\text{eff. rel. C-div. on } C_T \text{ of deg. } n\}$.

We have:

$$\begin{array}{ccc}
 C^n & \xrightarrow{P} & \text{Div}_C^n \\
 \pi \downarrow & \nearrow \bar{P} & \\
 C^{(n)} & &
 \end{array}
 \quad (P_1, \dots, P_n) \in C^n(T) \mapsto P_1(T) + \dots + P_n(T)$$

As Div_C^n not yet known to be a scheme, make \bar{P} "directly"!

Thm. \bar{P} is an isomorphism. (hence: Div_C^n is repr., we know $C^{(n)}$ as functor).

Rem. This is not so easy to prove at points where the divisor is not reduced.

For $C = \mathbb{P}_k^1$ it is easy, and instructive (see Andrea's notes).

The following weaker version is easier.

Let $\Delta \subset C^n$ be the closed subset $\bigcup_{i < j} p_{i,j}^{-1} \Delta$, where $\Delta \subset C \times_k C$ diag.

Let $\text{Div}_C^{n, \text{et}} \subset \text{Div}_C^n : T \mapsto \{D \in \text{Div}_C^n(T) : \forall \text{ geom. pt. } \text{Spec}(k) \rightarrow T, D_k \text{ reduced}\}$
 $= \{D \in \text{Div}_C^n(T) : D \rightarrow T \text{ is etale}\}$.

Then:

$$\begin{array}{ccc}
 C^n - \Delta & \xrightarrow{P} & \text{Div}_C^{n, \text{et}} \\
 \pi \downarrow & \nearrow \bar{P} & \\
 C^{(n)} - \Delta & &
 \end{array}$$

Sketch of proof: produce the inverse of \bar{P} . For $T \rightarrow \text{Spec}(k)$ and $D \in \text{Div}_C^{n, \text{et}}(T)$, $D \rightarrow T$ loc. isom. to $\coprod_{1 \leq i \leq n} T$.
 etale

2. $\text{Pic}_{C/k}$ is representable if $C(k) \neq \emptyset$. Ref. deagrg.html, §10

Situation: add: C/k projective, geom. irreducible, given $P \in C(k)$.

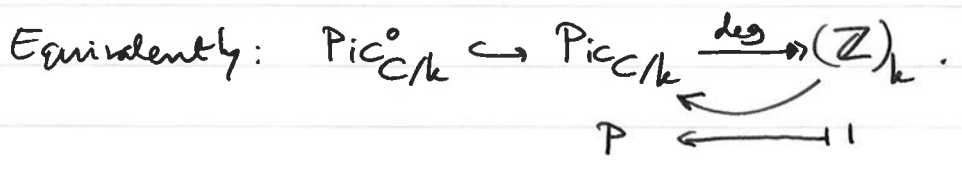
Recall from Nikitas's talk:

Def. $\text{Pic}_{C/k} : \text{Sch}/k \rightarrow \text{Ab}$, $T \mapsto \text{Pic}(C_T)/\text{Pic}(T) =$
 $= \{ \text{isom. classes of } (\mathcal{L}, r) : \mathcal{L} \text{ inv. } \mathcal{O}_{C_T}\text{-module, } r : \mathcal{O}_T \xrightarrow{\sim} \mathbb{P}_T^* \mathcal{L} \}$

Rem. For T/k , \mathcal{L} inv. \mathcal{O}_{C_T} -module, $T \xrightarrow{\chi(\mathcal{L})} \mathbb{Z}$, $\epsilon \mapsto \chi(\mathcal{L}|_{C_\epsilon})$ is
 loc. constant (flatness). Hence we can define

$\text{deg } \mathcal{L} : T \rightarrow \mathbb{Z}$, $\epsilon \mapsto \chi(\mathcal{L}) + g - 1$, where $g = \text{genus of } C = \dim_k H^1(C, \mathcal{O}_C)$.

Then $T = \coprod_{n \in \mathbb{Z}} \text{deg}(\mathcal{L})^{-1}(n)$. Hence $\text{Pic}_{C/k} = \coprod_{n \in \mathbb{Z}} \text{Pic}_{C/k}^n$.



Thm. a. $\text{Pic}_{C/k}$ is representable (hence every $\text{Pic}_{C/k}^n$ is, all $\cong \text{Pic}_{C/k}^0$)

b. $\text{Pic}_{C/k}^0$ is projective, $J := \text{Pic}_{C/k}^0$ (jacobian variety).

c. $C \xrightarrow{i} J$, $Q \mapsto \mathcal{O}(Q-P)$ induces $\text{Pic}_{J/k} \xrightarrow{i^*} \text{Pic}_{C/k}$

conn. comp. of 0 $\cong \text{Pic}_{J/k}^0 \xrightarrow{\nu} J$ the natural principal polarisation of J .

Main idea of proof of a. $\text{Div}_C^g \xrightarrow{\varphi} \text{Pic}_{C/k}^g$, $D \mapsto [\mathcal{O}_{C_T}(D)]$ is birational
 \downarrow
 $C^{(g)}_{-\Delta} = \text{Div}_C^{\text{set}}$

Hand waving style: for \mathcal{L} in $\text{Pic}^g(C_k)$, $\varphi^{-1}(\mathcal{L}) = \frac{H^0(C_k, \mathcal{L}) - \text{sol}}{k^*}$,
 so φ induces an open immersion on the open set U of $C^{(g)}$ where $h^0(C_k, \mathcal{L}) = 1$.

More seriously: $\forall T/k, \forall \mathcal{L}$ in $\text{Pic}^g(C_T)$: $\begin{matrix} C_T & \mathcal{L} \\ f_* \downarrow & \\ T & (R^1 f_*) \mathcal{L} \end{matrix}$ is coherent,

let $U := T - \text{supp } (R^1 f_*) \mathcal{L}$, then $f_* \mathcal{L}|_{C_U}$ is loc. free rank 1, its formation commutes with base change, $\exists!$ ~~divisor~~ D in $\text{Div}_C^g(U)$ s.t. $\mathcal{L}|_{C_U} \cong \mathcal{O}_{C_U}(D)$ mod $\text{Pic}(U)$.

Applying this to $C^{(g)}$ this gives a dense open $U \subset C^{(g)}$ on which φ induces an open immersion: $U \xrightarrow{\varphi} \text{Pic}_{C/k}^g$, for all $T \rightarrow \text{Pic}_{C/k}^g$ the pullback of φ exists $V \hookrightarrow T$ (V is a scheme) and is op.imm.

Now assume $k = \bar{k}$. Then the translates $t_x \circ \varphi$, $x \in \text{Pic}(C)$, cover $\text{Pic}_{C/k}^g$, and $\text{Pic}_{C/k}^g$ is represented by the scheme obtained by gluing these charts. See §10.4.

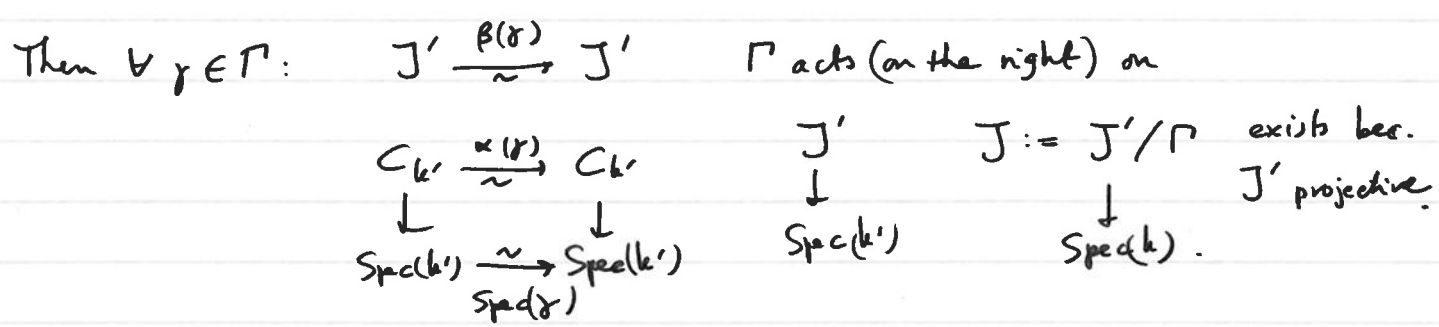
$C^{(g)} \rightarrow \text{Pic}_{C/k}^g$ is surjective, hence $\text{Pic}_{C/k}^g$ is proper.

Hence $\text{Pic}_{C/k}^g$ is proper, hence an ab. var., hence projective. (Add!)

Rem. Θ -divisor in $\text{Pic}_{C/k}^{g-1}$: " $\{\mathcal{L} \in \text{Pic}^{g-1}(C) : h^0(\mathcal{L}) \geq 1\}$ ". Ample!

But what if $k \neq \bar{k}$?

Then take $k \rightarrow k'$ finite Galois, gr. Γ , s.t. $\text{Pic}_{C_{k'}/k'}^g$ repr., say by J' over k' .



This is called Galois descent of schemes.

Question: what does J represent?

Answer: if $C(k) \neq \emptyset$, then $\text{Pic}_{C/k}^g : T \mapsto \text{Pic}(C_T) / \text{Pic}(T)$.

Example $k := \mathbb{R}$, $C = V(x^2 + y^2 + z^2) \subset \mathbb{P}_{\mathbb{R}}^2$. Note $C(\mathbb{R}) = \emptyset$, $C_{\mathbb{C}} \cong \mathbb{P}_{\mathbb{C}}^1$.

Then $\text{Pic}_{C/k}$ is not a sheaf for the étale topology:

$$\text{Pic}_{C/\mathbb{R}}(\mathbb{B}) = \text{Pic}(C_{\mathbb{C}}) \xrightarrow[\sim]{\text{deg}} \mathbb{Z}, \text{ with trivial } \text{Gal}(\mathbb{C}/\mathbb{R})\text{-action.}$$

$$\text{Pic}_{C/\mathbb{R}}(\mathbb{R}) = \text{Pic}(C) \xrightarrow[\sim]{\text{deg}} 2\mathbb{Z}.$$

So, in general, one should define $\text{Pic}_{C/k}$ as the ^{étale} sheafification of $T \mapsto \text{Pic}(C_T) / \text{Pic}(T)$: $f: C \rightarrow \text{Spec } k$, $(R^1 f_*) \mathcal{G}_m$.

$$f_* : \underset{\text{Ab}}{\uparrow} (\text{Sch}/C)_{\text{ét}} \rightarrow \underset{\text{Ab}}{\uparrow} (\text{Sch}/k)_{\text{ét}}$$

3. fpqc - descent of modules. Ref: {MSc thesis Zomervucht, Vistoli's notes descent.pdf, § 4.2}

We first consider the situation in Set. $(X \rightarrow S) \mapsto (X_T \rightarrow T)$

Let $T \xrightarrow{f} S$ be surjective (in Set). Gives $f^{-1}: \text{Set}/S \rightarrow \text{Set}/T$.

Then, for $t_1, t_2 \in T$ with $f(t_1) = f(t_2)$, $(X_T)_{t_1} = X_{f(t_1)} = X_{f(t_2)} = (X_T)_{t_2}$.

Moreover, for $s \in S$, $t_1, t_2, t_3 \in f^{-1}(s)$, $(X_T)_{t_1} = (X_T)_{t_2} = (X_T)_{t_3}$, compatible (given)

Now given $Y \rightarrow T$ with $\forall s \in S, \forall t_1, t_2 \in f^{-1}(s): Y_{t_1} \xrightarrow{\sim} Y_{t_2}$

such that $\forall s \in S, \forall t_1, t_2, t_3 \in f^{-1}(s): Y_{t_1, t_3} = Y_{t_2, t_3} \circ Y_{t_1, t_2}$, we get an $X \rightarrow S$ + $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & \square & \downarrow \\ T & \xrightarrow{f} & S \end{array}$, and these determine $X \downarrow S$ up to unique isomorphism.

(To make such an X/S , choose a point in every fibre of f , i.e., a section of f , or take a suitable limit.)

with the same image in $S(U)$ (resp. $S(V)$)

Reformulation without using elements of sets:

$\forall U, \forall t_1, t_2: U \rightarrow T$, we want $\varphi_{t_1, t_2}: t_1^* Y \xrightarrow{\sim} t_2^* Y$ s.t.

$\forall U, \forall t_1, t_2, t_3: U \rightarrow T: \varphi_{t_1, t_3} = \varphi_{t_2, t_3} \circ \varphi_{t_1, t_2}$.

(Rem. universal case: $U = T \times_S T, t_i = \text{pr}_i; V = T \times_S T \times_S T, t_i = \text{pr}_i$.)

Now let $T \xrightarrow{f} S$ be an fpqc cover in Sch, $Y \rightarrow T$.

A descent datum for $Y \rightarrow T$ rel. to $T \xrightarrow{f} S$ is:

an isom. $\varphi: \text{pr}_1^{-1} Y \xrightarrow{\sim} \text{pr}_2^{-1} Y$ on $T \times_S T$, such that on $T \times_S T \times_S T: (\text{pr}_1, \text{pr}_3)^{-1} \varphi = (\text{pr}_2, \text{pr}_3)^{-1} \varphi \circ (\text{pr}_1, \text{pr}_2)^{-1} \varphi$.

Then we have $\text{Sch}/S \xrightarrow{f^{-1}} \text{Sch}/T$ + desc.dat., fully faithful (by Pol's lecture). Unfortunately not always essentially surjective.

But f^{-1} is ess. surjective when restricted to schemes that are affine/S, resp. /T. To prove this, one treats the analogous case of quasi-coh. \mathcal{O} -modules. The reduction to S and T affine is easy.

So now $A := \mathcal{O}(S), B := \mathcal{O}(T), A \xrightarrow{f^*} B$ faithfully flat.

Let M be a B -module and φ a descent datum on M to A :

$\varphi: \text{pr}_1^* M \xrightarrow{\sim} \text{pr}_2^* M$ isom. of $B \otimes_A B$ -modules, $\text{pr}_i^*: B \rightarrow B \otimes_A B, b \mapsto b \otimes 1$

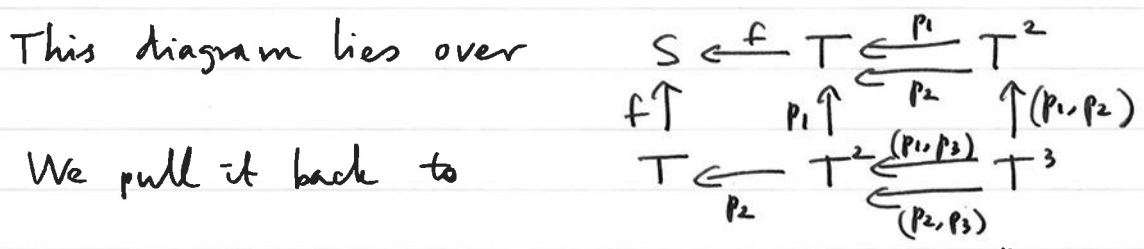
$p_2^* : B \rightarrow B \otimes_A B, b \mapsto 1 \otimes b$. Then $(p_1, p_3)^* \varphi = (p_2, p_3)^* \varphi \circ (p_1, p_2)^* \varphi$,
 isomorphisms of $B^{\otimes 3}$ -modules (\otimes over A). the $B^{\otimes 2}$ -module

Thm (Grothendieck, I presume). Let $N = \{m \in M : \varphi(p_1^* m) = p_2^* m \text{ in } p_2^* M\}$.
 Then N is a sub- A -module of M .

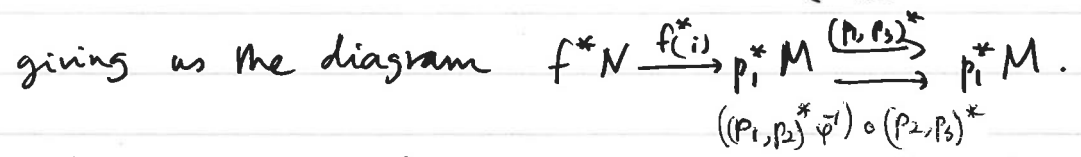
The B -module morphism $B \otimes_A N \rightarrow M, b \otimes n \mapsto b \cdot n$, is an isomorphism,
 compatible with descent data: pull-back on LHS, φ on RHS.

Proof. (I follow Zornvacht, but written using projection maps, not tensors...).

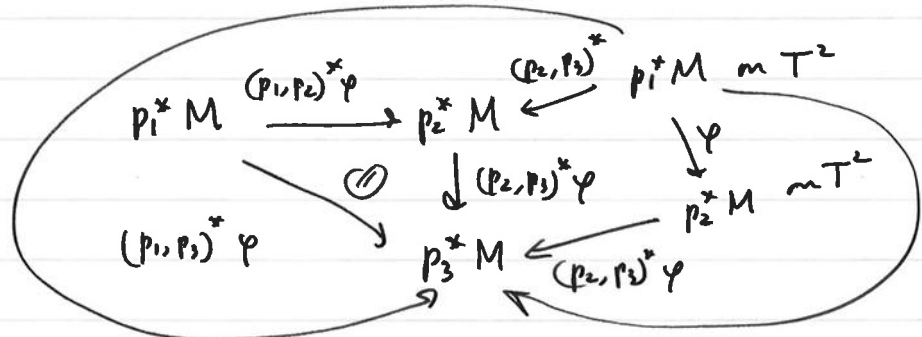
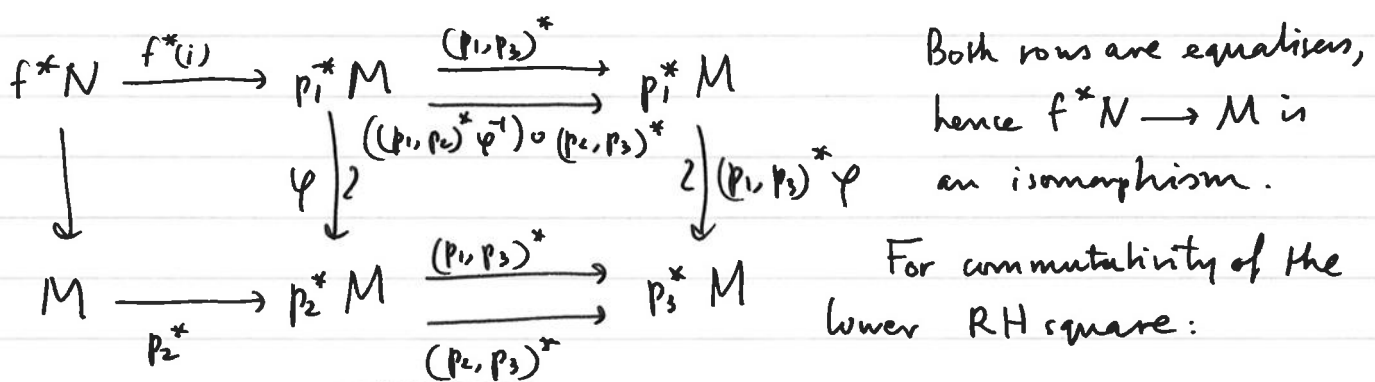
We have the equaliser diagram $N \xrightarrow{i} M \xrightarrow[p_1^* \varphi]{p_1^*} p_1^* M$



We pull it back to



This last diagram is isomorphic to the equaliser diagram from Pol's lecture obtained from viewing M as an A -module:



6.

Thm. Let X be a scheme. Then the natural morphisms of groups $H^i(X_{\text{ét}}, \mathbb{G}_m) \rightarrow H^i(\text{Sch}/X)_{\text{ét}}, \mathbb{G}_m \rightarrow H^i(\text{Sch}/X)_{\text{fpqc}}, \mathbb{G}_m$ are isomorphisms.

Proof. In all cases, $\mathbb{G}_m = \text{Aut}(\mathbb{A}^1_{\mathbb{Z}})$ that \mathbb{G}_m and $\mathbb{A}^1_{\mathbb{Z}}$ are sheaves was explained in Pol's lecture. By Lurie's lecture: these H^i 's are the sets of isomorphism classes of twists of $\mathbb{A}^1_{\mathbb{Z}}$: the $\mathbb{A}^1_{\mathbb{Z}}$ -modules \mathcal{L} that are locally (for the corresponding topology) isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$. We claim that such \mathcal{L} are all already Zariski-locally isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$.

Let \mathcal{L} be an $\mathbb{A}^1_{\mathbb{Z}}$ -module on $(\text{Sch}/X)_{\text{fpqc}}$ that is locally isom. to $\mathbb{A}^1_{\mathbb{Z}}$.

We want to show that $\mathcal{L}|_{X_{\text{zar}}}$ is locally isom. to $\mathbb{A}^1_{\mathbb{Z}}$.

We may (and do) assume that X is affine, $X = \text{Spec } A$. Then there is

$A \rightarrow B$ faithfully flat such that $f^*\mathcal{L}$ is isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$. Pick an

$X \in \mathbb{A}^1_{\mathbb{Z}}$ isomorphism $\alpha: \mathbb{A}^1_{\mathbb{Z}} \xrightarrow{\sim} f^*\mathcal{L}$. The natural descent datum

of \mathcal{L} rel. to $Y \rightarrow X$ gives a descent datum on $\mathbb{A}^1_{\mathbb{Z}}$ rel. to $Y \rightarrow X$.

Hence, by descent of quasi-coherent $\mathbb{A}^1_{\mathbb{Z}}$ -modules, $\mathcal{L}|_{X_{\text{zar}}}$ is quasi-coherent, and loc. free of rank 1. \square