TAG, 2016/05/18, Bas Edixhoven.

The proper base change theorem, following SP 0956.

The theorem let
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$
be cartesian, in Sch, with f proper, let $\mathcal{F}$ be a torsion abelian sheaf on X_{et}, then $\forall p$, $g^{-1}(R^pf_*)\mathcal{F} \longrightarrow (R^pf'_*)(g'^{-1}\mathcal{F})$ is isomorphism.

Terminology: f commutes with cohomology if $\forall\ Y' \xrightarrow{g} Y$, $\forall \mathcal{F}$, $\forall p$, the base change map is an isomorphism.

Important reduction step: let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f & g proper.

If f & g commute with cohomology, then so does $g\circ f$.

Proof. Let $h\colon Z' \to Z$ in Sch and consider:
$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "h"] & Z \end{tikzcd}$$
$(\mathcal{F} = \operatorname{colim}_n \mathcal{F}[n])$ $\downarrow$ $\mathcal{F}$ torsion abelian on X_{et}.

Here I do not follow SP.

We have: (1) $\forall p$: $(h')^{-1}((R^pf_*)\mathcal{F}) \xrightarrow{\sim} (R^pf'_*)((h'')^{-1}\mathcal{F})$

(2) $\forall q$ $\forall \mathcal{G}$ tors.ab. on Y:
$$h^{-1}((R^qg_*)\mathcal{G}) \xrightarrow{\sim} (R^qg'_*)((h')^{-1}\mathcal{G}).$$

(3) Leray sp. seq. for $g\circ f$:
$$E_2^{p,q} = (R^pg_*)((R^qf_*)\mathcal{F}) \Longrightarrow (R^{p+q}(g\circ f)_*)\mathcal{F} \quad \text{in } Ab(Z_{et}).$$
Applying h^{-1} (exact!) gives a spectral sequence in $Ab(Z'_{et})$:
$$h^{-1}E_2^{p,q} = h^{-1}(R^pg_*)(R^qf_*)\mathcal{F} \Longrightarrow h^{-1}(R^{p+q}(g\circ f)_*)\mathcal{F}$$

(4) Leray sp. seq. for $g'\circ f'$:
$$'E_2^{p,q} = (R^pg'_*)(R^qf'_*)((h'')^{-1}\mathcal{F}) \Longrightarrow (R^{p+q}(g'\circ f')_*)((h'')^{-1}\mathcal{F})$$

(5) Functoriality of Leray spectral sequence gives a morphism of spectral sequences:
$$\begin{tikzcd}[column sep=small] & h^{-1}E_2^{p,q} = h^{-1}(R^pg_*)(R^qf_*)\mathcal{F} \arrow[dl, "\sim"', "(2)"] \arrow[dd] \arrow[r, Rightarrow] & h^{-1}(R^{p+q}(g\circ f)_*)\mathcal{F} \arrow[dd] \\ (R^pg'_*)(h')^{-1}(R^qf_*)\mathcal{F} \arrow[dr, "\sim", "(1)"'] & & \\ & {}'E_2^{p,q} = (R^pg'_*)(R^qf'_*)(h'')^{-1}\mathcal{F} \arrow[r, Rightarrow] & (R^{p+q}(g'\circ f')_*)(h'')^{-1}\mathcal{F} \end{tikzcd}$$

So this morphism is an isomorphism on page 2, but then it is an isomorphism on all pages $r \geqslant 2$, hence on $Gr^\bullet h^{-1}(R^n(g\circ f)_*)\mathcal{F} \to Gr^\bullet(R^n(g'\circ f')_*)((h'')^{-1}\mathcal{F})$

hence also on the filtered objects $h^{-1}(R^n(g\circ f)_*)\mathcal{F} \longrightarrow (R^n(g'\circ f')_*)((h'')^{-1}\mathcal{F})$.

(think of $\begin{array}{|c|c|}\hline * & * \\ \hline 0 & * \\ \hline\end{array}$). $\boxtimes$.

To use this reduction via factorisation, we reduce to the case where f is projective. But for that we first reformulat "f commutes with cohom.".

I do not see the necessity of reducing to $\mathbb{Z}/\ell\mathbb{Z}$-modules.

Lemma (0A4B)(half of it) Let $f: X \to Y$ in Sch be proper.
Then TFAE:
1. f commutes with cohomology
2. $\forall$ injective torsion abelian sheaf I on X_{et}, $\forall g: Y' \to Y$ $\forall q > 0$, $(R^q f'_*)((g')^{-1} I) = 0$ (where
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$
).

Proof $1 \Rightarrow 2$: assuming 1, $(R^q f'_*)((g')^{-1}I) = g^{-1}(R^q f_*)I$, then use that I is injective.

$2 \Rightarrow 1$. Assume 2. Let $\mathcal{F}$ in $Ab(X_{et})$ be torsion. Let $\mathcal{F} \to I^\bullet$ be an inj. torsion resolution. Then $(g')^{-1}\mathcal{F} \to (g')^{-1}I^\bullet$ is a f'_*-acyclic resolution (by 2) hence $\forall q \geq 0$, $(R^q f'_*)(g^{-1}\mathcal{F}) = H^q(f'_*(g')^{-1}(I^\bullet)) = H^q(g^{-1} f_* I^\bullet) =$
(by lemma 50.76.5 (0A3U), that we will leave as it is, here).
$= g^{-1} H^q(f_* I^\bullet) = g^{-1}(R^q f_*)\mathcal{F}$. $\boxtimes$

<u>Reduction from torsion ab. sheaves to $(\mathbb{Z}/\ell\mathbb{Z})_{X_{et}}$-modules</u>.

For $\mathcal{F}$ in $Ab(X_{et})$ torsion: $\mathcal{F} = \operatorname{colim}_n \mathcal{F}[n]$, filtered colimit, with which $R^q f_*$, $(g')^{-1}$ etc commute. So may assume $\mathcal{F}$ in $(\mathbb{Z}/n\mathbb{Z})_{X_{et}}$ for some $n \geq 1$. If $n > 1$ not prime: $\ell \cdot n' = n$. $\mathcal{F}[\ell] \hookrightarrow \mathcal{F} \twoheadrightarrow \underbrace{\mathcal{F}/\mathcal{F}[\ell]}$ killed by n', then long exact sequences and 5-lemma $\boxtimes$.

Now comes the 2nd interesting reduction.

Lemma 50.76.7 (0A4C) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ in Sch with f, g proper, and f surjective. If f and $g \circ f$ commute with cohomology, then g too.

Proof. We use the previous reduction and the previous lemma. Let ℓ be prime, and I in $(\mathbb{Z}/\ell\mathbb{Z})_{X_{et}}$ injective. Let $h : Z \to Z'$, etc.

$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] & I \\ Z' \arrow[r, "h"] & Z \end{tikzcd}$$

To prove: $(h')^{-1} I$ is g'_* acyclic.

Choose $f^{-1} I \hookrightarrow J$ with J in $(\mathbb{Z}/\ell\mathbb{Z})_{X_{et}}$ injective.

Then $f_* f^{-1} I \hookrightarrow f_* J$ and $I \hookrightarrow f_* f^{-1} I$ bec. f surjective (use stalks at geom. pts), so $I \hookrightarrow f_* J$, splits bec. I injective.

So it suffices to show that $(h')^{-1} f_* J$ is g'_*-acyclic.

As f commutes with cohom., $(h')^{-1}(f_* J) = f'_*((h'')^{-1} J)$.

As f and $g \circ f$ commute with cohomology, $\forall q > 0$ $(R^q f'_*)(h'')^{-1} J = 0$ and $(R^q (g' \circ f')_*)((h'')^{-1} J) = 0$. Now we use the Leray spectral sequence for $g' \circ f'$:

$$E_2^{p,q} = (R^p g'_*)\underbrace{(R^q f'_*)((h'')^{-1} J)}_{\substack{\| \\ 0 \text{ for } q>0}} \Longrightarrow \underbrace{(R^{p+q} (g' \circ f')_*)((h'')^{-1} J)}_{\substack{\| \\ 0 \text{ for } p+q>0}}$$

$$\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ g'_*(f'_*(h'')^{-1} J) & (R^1 g'_*)(\quad) & (R^2 g'_*)(\quad) & \cdots \\ & \| & \| & \cdots \\ & 0 & 0 & \end{array}$$

$\boxtimes$.

Reduction to $X \xrightarrow{f} Y$ projective. Let $X \xrightarrow{f} Y$ be proper.

Then (limits and Chow's lemma) $\exists$

$$\begin{tikzcd} X \arrow[dr, "f"'] & X' \arrow[l, "\pi"', "\text{proper and surjective}"] \arrow[r, hook, "\text{closed immersion}"] \arrow[d] & \mathbb{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

Closed immersions commute with cohom.

So we are reduced to $\mathbb{P}^n_Y \xrightarrow{f} Y$.

Then use (ex. 8 Andrea's talk):

$$\begin{tikzcd} & \mathbb{P}^n_Y \arrow[r, "f"] & Y \\ (\mathbb{P}^1_Y)^n \arrow[ur, two heads] \arrow[urr] \arrow[r] & (\mathbb{P}^1_Y)^{n-1} \arrow[r] & \cdots \ \mathbb{P}^1_Y \arrow[r] & Y \end{tikzcd}$$

all relative $\mathbb{P}^1$'s.

So, reduced to $\mathbb{P}^1_S \to S$.

That follows from Nikita's lemmas.