TAG, 2016/05/25 Bas Edixhoven.

Compactly supported cohomology, following SGA 4.5, 47–53.

Let k be a field, X/k separated, of finite type. By Nagata, $\exists\, j : X \hookrightarrow \overline{X}$, an open immersion in a proper k-scheme. Then (as for any open immersion) $j^{-1} : Ab(\overline{X}_{et})_{tors} \to Ab(X_{et})_{tors}$ has a left adjoint $j_! : Ab(X_{et})_{tors} \to Ab(\overline{X}_{et})_{tors}$ called extension by zero. Let $Z := \overline{X} - X$ and $i : Z \hookrightarrow \overline{X}$ the closed immersion. Then for $\mathcal{F}$ in $Ab(X_{et})_{tors}$, $j_!\mathcal{F}$ is uniquely determined by: $j^{-1}j_!\mathcal{F} = \mathcal{F}$, $i^{-1}j_!\mathcal{F} = 0$. The compactly supported cohomology $H^q_c(X_{et}, \mathcal{F})$ is defined as: $H^q_c(X_{et}, \mathcal{F}) := H^q(\overline{X}_{et}, j_!\mathcal{F})$; indeed, this is shown to be independent of the choice of compactification j, using the proper base change theorem. Note that if X/k is proper, then $H^q_c(X_{et}, \mathcal{F}) = H^q(X_{et}, \mathcal{F})$.

A very important application is this. For X, $\mathcal{F}$ as above, $U \subset X$ open and $Y = X - U$, we have the long exact sequence obtained by taking cohom. on $\overline{X}_{et}$ of the s.e.s. $j_{U!}\mathcal{F}|_U \hookrightarrow \mathcal{F} \twoheadrightarrow i_{Y*}i_Y^{-1}\mathcal{F}$ extended by 0 ($j_!$ is exact

$$\cdots \longrightarrow H^q_c(U, \mathcal{F}|_U) \longrightarrow H^q_c(X, \mathcal{F}) \longrightarrow H^q_c(Y, \mathcal{F}|_Y) \longrightarrow H^{q+1}_c(U, \mathcal{F}_U) \longrightarrow \cdots$$

Relative version. Let $f : X \to S$ separated, of finite type, with X and S noetherian. Then $\exists$

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \\ & S & \end{tikzcd}$$

with $\overline{f}$ proper & j an open immersion

For $\mathcal{F}$ in $Ab(X_{et})_{tors}$ one defines $(Rf_!)\mathcal{F} = (R\overline{f}_*)(j_!\mathcal{F})$ ($= \overline{f}_* I^\cdot$ where $j_!\mathcal{F} \to I^\cdot$ is an injective resolution); this is indep. of the choice of j, name: derived proper push forward.

The proper base change theorem implies: $\forall$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

$\cancel{g^{-1}(Rf_!)}$ $g^{-1}(Rf_!)\mathcal{F} = (Rf'_!)(g')^{-1}\mathcal{F}$

Thm. Let S be noetherian, $f : X \to S$ separated, of finite type, $n \in \mathbb{N}$ s.t. $\forall s \in S$: $\dim(X_s) \leq n$, $\mathcal{F}$ in $Ab(X_{et})_{tors}$. Then $\forall\, q > 2n$: $(R^qf_!)\mathcal{F} =$
(In other words: $(Rf_!)\mathcal{F}$ is in $D(Ab(S_{et})_{tors})^{[0,2n]}$.)
If $\mathcal{F}$ is constructible then so are the $(R^qf_!)\mathcal{F}$ (in other words: $(Rf_!)\mathcal{F}$ is in $D^b_c(Ab(S_{et})_{tors})$: complexes with bounded below, constructible cohomology.)

We prove the 1st claim, by induction on n.

For $n=0$: $\forall\, \bar{s} \to S$ geom pt and $\forall\, q>0$: $((R^q f_!)F)_{\bar{s}} = H^q_c(X_{\bar{s}}, F_{\bar{s}}) =$

$= H^q_c(X_{\bar{s},red}, F_{\bar{s}}) = 0$ bec. $X_{\bar{s},red}$ = finite union of copies of $\bar{s}$.

Now let $n>0$ and assume that $\forall\, Y \xrightarrow{g} T$ with fibers of dim $\leq n-1$ and $\forall\, q > 2(n-1)$, and $\forall\, \mathcal{G}$ on Y, $(R^q g_!)\mathcal{G} = 0$.

Let $X \xrightarrow{f} S$, F on X, st. $\forall\, s \in S$, $\dim(X_s) \leq n$. To prove: $\forall\, q>2n$, $(R^q f_!)F =$

It suffices to prove that $\forall\, \bar{s} \to S$, $\forall\, q > 2n$: $H^q_c(X_{\bar{s}}, F_{\bar{s}}) = 0$.

So let $k = \bar{k}$, X/k finite type, separated, $\dim(X) \leq n$, F on X_{et}

To show: $\forall\, q>2n$: $H^q_c(X,F) = 0$. We may assume X is reduced.

Let $x_1, \dots, x_r$ be the generic pts of the irred. components of X.

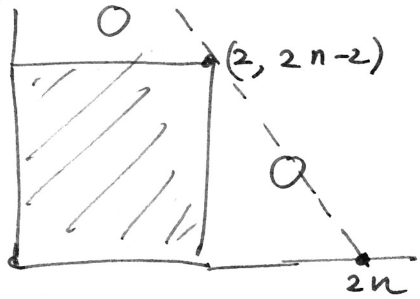
Let $f_i \in \mathcal{O}_{X,x_i}$ be non-constant whenever $\dim \overline{\{x_i\}} > 0$.

Let $U \subset X$ be open, containing all x_i, and $f \in \mathcal{O}_X(U)$ s.t. $\forall i$: $f_{x_i} = f_i$.

Then $f : U \to \mathbb{A}^1_k$ has its fibres of dim. $\leq n-1$, and $Z := X - U$ has dim. $\leq n-1$.

Consider the Leray spectral sequence:

$$E_2^{p,q} = H^p_c(\mathbb{A}^1_k, (R^q f_!)F|_U) \Rightarrow H^{p+q}_c(U,F)$$



So indeed, $\forall\, m>2n$, $H^m_c(U,F) = 0$.

$$\underset{\overset{\parallel}{0}}{H^m_c(U,F)} \to H^m_c(X,F) \to \underset{\overset{\parallel}{0}}{H^m_c(Z,F)}.$$

$\boxtimes$

<u>Exercise</u>. Let $k = \bar{k}$, X/k separated, of finite type, irreducible, $d := \dim(X)$. Then $\forall\, n \geq 1$, inv. in k, $H^{2d}_c(X, \mu_n^{\otimes d}) = \mathbb{Z}/n\mathbb{Z}$.

Note: $H^0(X, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, this is a hint at Poincaré duality.