TAG, 2016/05/25 Bas Edixhoven

ℓ-adic cohomology, Lefschetz trace formula, Weil conjectures.

First the case of constant sheaves $\mathbb{Q}_\ell$; this is well explained in Appendix C of Hartshorne's book AG (well, for smooth proper schemes $/k = \bar{k}$

Let $k = \bar{k}$, ℓ prime, $\ell \neq 0$ in k. Let X/k separated, finite type.
Then $H^i(X_{et}, \mathbb{Q}_\ell) := \mathbb{Q} \otimes \left(\lim\limits_{n} H^i(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z}) \right)$; finite dim. $\mathbb{Q}_\ell$-vect-space, zero if $i > 2\dim(X)$.
Same for $H^i_c(X, \mathbb{Q}_\ell)$.

Now let k be a finite field, $q := \#k$, $q = p^r$, p prime. $k \to \bar{k}$ an algebraic closure.
Let X/k separated, of finite type.
Let $F_{p,X} : X \to X$ be the absolute p-Frobenius: on every open affine U of X it corresponds to the p-power-map on $\mathcal{O}(U)$. Equivalently: $F_{p,X}$ is the identity on the top. space X, and $\forall\, U \subset X$ open, $F^*_{p,X}(U) : \mathcal{O}(U) \to \mathcal{O}(U)$, $f \mapsto f^p$.

Then
$$\begin{tikzcd} X \arrow[r, "F_{p,X}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "\sim", "F_{p,\mathrm{Spec}(k)}"'] & \mathrm{Spec}(k) \end{tikzcd}$$
. Let $F_X := F^r_{p,X}$ = abs. q-Frobenius of X.
Then
$$\begin{tikzcd} X \arrow[rr, "F_X"] \arrow[dr] & \circlearrowleft & X \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

If $X = Z(f) \hookrightarrow \mathbb{A}^n_k$, then $(F_X)_{\bar{k}} : (a_1, \ldots, a_n) \mapsto (a_1^q, \ldots, a_n^q)$.

Base change to $\bar{k}$ gives:
$$\begin{tikzcd} X_{\bar{k}} \arrow[rr, "(F_X)_{\bar{k}}"] \arrow[dr] & \circlearrowleft & X_{\bar{k}} \arrow[dl] \\ & \mathrm{Spec}(\bar{k}) & \end{tikzcd}$$

Note: $X_{\bar{k}}(\bar{k})^{(F_X)_{\bar{k}}} = X(k)$. In fact, $(X_{\bar{k}})^{(F_X)_{\bar{k}}} = \coprod_{x \in X(k)} \mathrm{Spec}(k)$.

LTF: $\# X(k) = \sum_i (-1)^i \, \mathrm{Tr}\left((F_X)^*_{\bar{k}}, H^i_c(X_{\bar{k}}, \mathbb{Q}_\ell) \right)$.

TAG 03V2

Weil conjectures: (proven by Dwork, Grothendieck + Artin, Deligne)
For X/k proper & smooth, $\det\left(1 - (F_X)^*_{\bar{k}} \cdot T, H^i(X_{\bar{k}}, \mathbb{Q}_\ell)\right) \in \mathbb{Q}_\ell[T]$ has coeff. in $\mathbb{Z}$ and the complex roots have abs. value $q^{-i/2}$.

Actually, the Weil conjectures are about the zeta-function of X, and what is above is the cohomological explanation. Weil had conjectured (or suggested) the existence of a cohomology theory.

More general LTF. Now let $\mathcal{F}$ be a (constructible) $\mathbb{Q}_\ell$-sheaf on X (see TAG 03UL). Then: (TAG 03V2)

$$\sum_{x \in X(k)} \mathrm{Tr}\left((\pi_x)^*_{\bar{k}}, \mathcal{F}_{\bar{x}}\right) = \sum_i (-1)^i \, \mathrm{Tr}\left((F_X)^*_{\bar{k}}, H^i_c(X_{\bar{k}}, \mathcal{F})\right).$$

RHS = Euler char. of $(F_X)^*_{\bar{k}}$ on cohom. of $\mathcal{F}$ on $X_{\bar{k}}$,

LHS = same but on $X_{\bar{k}}^{(F_X)_{\bar{k}}}$.

You wouldn't immediately see it, but this more general LTF can easily be reduced to the case $\dim(X) \leq 1$.