SPECTRAL SEQUENCES

I DON'T CLAIM ANY ORIGINALITY FOR THESE NOTES, EVERYTHING CAN BE FOUND IN [WEIBEL, AN INTRODUCTION TO HOMOLOGICAL ALGEBRA]. I TRIED JUST TO SIMPLIFY WHERE POSSIBLE.

QUESTION

LET $\mathcal{A}, \mathcal{B}, \mathcal{C}$ BE AB. CAT.'S, $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ BE HALF EXACT FUNCTORS, SAY LEFT; WHAT IS THE RELATION BETWEEN THE DERIVED FUNCTORS

$$R^i G \circ R^i F \xrightarrow{\ ?\ } R^i (G \circ F) \qquad ?$$

~~CAN YOU~~ WHEN G IS EXACT? $\quad R^i(G\circ F) = G \circ R^i F$

Thm · Grothendieck

LET $\mathcal{A}, \mathcal{B}, \mathcal{C}$ BE AB. CAT.; $\mathcal{A}, \mathcal{B}$ WITH ENOUGH INJECT.; $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ LEFT EXACT FUNCTORS.

IF F SENDS INJECTIVES OBJECTS IN G-ACYCLIC OBJECTS, THEN THERE EXIST A (1^{st} Q.) SPECTRAL SEQUENCE S.T.

$$E_2^{p,q} = (R^p G)(R^q F)(A) \Longrightarrow R^{p+q}(GF)A$$

LOOKING AT THE LOW DEGREE TERMS, WE HAVE:

$$0 \to (R^1 G)\, FA \to R^1(GF)A \to G(R^1F(A)) \to (R^2G)(FA) \to R^2(FG)(A)$$

Review OF HOMOM. ALG.

$\mathcal{A}, \mathcal{B}$ AB. CAT, $\mathcal{A}$ HAS ENOUGH INJECTIVES, $\mathcal{A} \xrightarrow{F} \mathcal{B}$ L.E. FUNCTOR

WE ~~CAN~~ GET $\{R^i F\}_{\mathbb{N}} : \mathcal{A} \to \mathcal{B}$ UNIVERSAL δ-FUNCTOR.

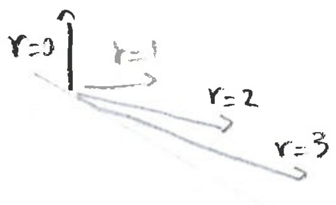
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & Ch(\mathcal{A}) \arrow[d] \\ \mathcal{B} & Ch(\mathcal{B}) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, mapsto] & (A \to 0) \arrow[r, phantom, "\overset{\text{QUASI-ISOMORPHISM}}{\sim}"] & (I^\bullet) \arrow[d] \\ & & F(I^\bullet) \end{tikzcd}$$

$$R^pFA := h^p(FI^\bullet).$$

Def

$\mathcal{A}$ AB. CAT, A (1^{st} QUADRANT) SPECTRAL SEQUENCE CONSISTS OF

- $E_r^{p,q} \in Ob(\mathcal{A})$ FOR $r \geq a$ AND S.T. $E_r^{p,q} = 0$ IF $p<0$ OR $q<0$
- MORPHISMS $E_r^{p,q} \xrightarrow{d} E_r^{p+r,q-r+1}$ S.T. $d^2 = 0$
- ISOMORPHISMS $h(E_r^{p,q}) \simeq E_{r+1}^{p,q}$



example
(1^{st} Q.)

"COMPUTE" THE HOMOLOGY OF THE TOTAL COMPLEX OF A DOUBLE COMPLEX

Def

IF $\exists\, r_0$ S.T. $\forall r > r_0$ $E_r^{p,q} = E_{r+1}^{p,q}$, THEN WE DEFINE $E_\infty^{p,q}$ AS THE STABLE VALUE

Rmk

IF ~~Space~~ THE S.S. ARE 1^{st} Q. THEN $[\, r > p \quad r-1 > q \,]$

$$E_\infty^{p,q} = E_R^{p,q} \quad \text{WHERE } R > p \text{ AND } R > q+1$$

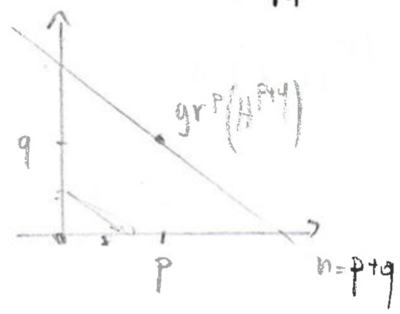
Def

LET $\{H^n\}_{\mathbb{N}}$ A FAMILY OF OBJECT IN $\mathcal{A}$, EACH OF THEM FILTERED

$(\text{I.E. } 0 = F^t H^n \subseteq \dots \subseteq F^s H^n = H^n \quad \text{WITH } t > s)$

WE SAY THAT A S.S. $E_r^{p,q}$ CONVERGES TO H^n, (NOTATION $E_r^{p,q} \Rightarrow H^n$) WHEN

$$gr^p(H^{p+q}) := F^p H^{p+q} / F^{p+1} H^{p+q} \simeq E_\infty^{p,q}$$



RMK

IN A 1^{st} Q. S.S. $t = n+1$ $s = 0$, THAT IS:

$$0 = F^{n+1} H^n \subseteq \dots \subseteq F^0 H^n = H^n$$

CONSTRUCTION Thm:

FOR EVERY FILTERED COMPLEX:

$$\begin{array}{ccl}
F^0 K^\bullet = K^\bullet & & K^0 \to K^1 \to K^2 \to K^3 \to \dots \\
\cup\!\!\mid & & \;\cup\!\!\mid \quad\; \cup\!\!\mid \quad\; \cup\!\!\mid \\
F^1 K^\bullet & & F^1K^0 \to F^1K^1 \to F^1K^2 \to \dots \\
\cup\!\!\mid & & \;\cup\!\!\mid \quad\; \cup\!\!\mid \quad\; \cup\!\!\mid \\
F^2 K^\bullet & & F^2K^0 \to F^1K^1 \to \;\to \dots \\
 & & \vdots
\end{array}$$

~~IF THE FILTRAT~~

IF THE FILTRATION IS CANONICALLY BOUNDED, I.E:

- $F^0(K^n) = K^n$
- $F^{n+1}(K^n) = 0$

WE CAN CONSTRUCT A SPECTRAL SEQUENCE:

$$E_1^{p,q} = H^{p+q}(gr^p K^\bullet) \Rightarrow H^{p+q}(K^\bullet)$$

Thm

FOR EVERY 1^{st} QUA DOUBLE COMPLEX $C^{\bullet,\bullet}$, WE HAVE TWO ~~SPES~~ S.S.

$$'E_2^{p,q} = H_h^p\left(H_v^q\left(C^{\bullet,\bullet}\right)\right) \Rightarrow H^{p+q}\left(\mathrm{Tot}\left(C^{\bullet,\bullet}\right)\right)$$

$$''E_2^{p,q} = H_v^p\left(H_h^q\left(C^{\bullet,\bullet}\right)\right) \Rightarrow H^{p+q}\left(\mathrm{Tot}\left(C^{\bullet,\bullet}\right)\right)$$

Def

LET $\mathcal{A}$ BE AN AB.CAT, LET $A^\bullet \in Ch(\mathcal{A})$. A CARTAN-EILEMBERG RESOLUTION OF $A^\bullet$ IS A DOUBLE COMPLEX $I^{\bullet,\bullet}$ OF INJECT. OBJS WITH A COAUGMENTATION MAP $A^\bullet \to I^{\bullet,\bullet}$ S.T. COBOUNDARIES AND COHOMOLOGY ARE INJECT. RES OF $B^p(A^\bullet)$ AND $H^p(A^\bullet)$:

$$\begin{tikzcd}
A^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "B^1(A)"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \cdots \\
I^{0,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "B(I^{1,0})"] & I^{1,0} \arrow[r] \arrow[d] & I^{2,0} \arrow[r] \arrow[d] & \cdots \\
I^{0,1} \arrow[r] \arrow[d] & I^{1,1} \arrow[r] & \cdots & \\
\vdots & & &
\end{tikzcd}$$

LEMMA

EVERY $A^\bullet \in Ch(\mathcal{A})$ HAS A C.-E. RESOLUTION.

Def

LET $\mathcal{A} \xrightarrow{F} \mathcal{B}$ BE A ~~RIGHT~~ Left EXACT FUNCTOR BETWEEN AB. CAT, $\mathcal{A}$ WITH ENOUGH INJECTIVES

WE DEFINE THE HYPER-DERIVED FUNCTOR OF F ~~AS~~:

$$\mathbb{R}^i F : Ch(\mathcal{A}) \longrightarrow \mathcal{B}$$

$$A^\bullet \longmapsto H^i\left(\mathrm{Tot}\ F(I^{\bullet,\bullet})\right) \quad \text{(ind. of choice of res.)}$$

WHERE $I^{\bullet,\bullet}$ IS A C.E. RES OF $A^\bullet$.

Thm

WE HAVE TWO SPECTRAL SEQUENCES:

$$'E_2^{p,q} = H^p\left(R^q F(A)\right) \Rightarrow \mathbb{R}^{p+q} F(A)$$

$$''E_2^{p,q} = \left(\mathbb{R}^p F\right)\left(H^q(A)\right) \Rightarrow \mathbb{R}^{p+q} F(A)$$

Thm - Grothendieck

Note about left exact functors

Usually, left exact functors are defined as follows:

Def

Let $\mathcal{A}$, $\mathcal{B}$ two abelian categories and $\mathcal{A} \xrightarrow{F} \mathcal{B}$ an additive functor.
F is said left exact when for every ~~exact seque~~

$$0 \to A \to B \to C \to 0$$

exact sequence in $\mathcal{A}$,

$$0 \to F(A) \to F(B) \to F(C)$$

is exact in $\mathcal{B}$.

Nevertheless, sometimes it is useful to give a definition for general categories.:

Def - II

Let C a category with finite projective limits. Then a functor $C \xrightarrow{F} C'$ is left exact if it commutes with finite projective limits.

(cf. SGA 4, I, Def 2.4.1)

In the following I would like to sketch why a left exact functor (in the sense of II) between Ab. Cat. is additive

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Now, in an abelian category $\mathcal{A}$, for every two objects A, B products and coproducts coincide and we have the diagram

$$\begin{tikzcd}
 & & A & & \\
A \arrow[urr, "1_A"] \arrow[rr, "i_A"] \arrow[drr, "0"'] & & A \oplus B \arrow[u, "p_A"'] \arrow[d, "p_B"] & & B \arrow[ll, "i_B"'] \arrow[ull, "0"'] \\
 & & B \arrow[urr, "1_B"'] & &
\end{tikzcd}$$

where i_A and i_B are the structure maps for the coproduct; while p_A and p_B for the product.

Moreover we can define a group law on the set $\mathrm{Hom}_{\mathcal{A}}(A,B)$. By the composition:

$$f+g := A \xrightarrow{(f,g)} B \oplus B \xrightarrow{\nabla = \binom{1_B}{1_B}} B$$

where $f,g \in \mathrm{Hom}(A,B)$.

→ or recovers (depending on your Def of abelian category)

One can check that this defines an additive structure on the category $\mathcal{A}$.

If we consider a left exact functor (in the sense of II- definition) $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between two abelian categories, then we have the ~~equality of~~ biproduct $\left(F(A) \oplus F(B), i_{F(A)}, i_{F(B)}, p_{F(A)}, p_{F(B)}\right)$ coincides with:

$$\left(F(A \oplus B), F(i_A), F(i_B), F(p_A), F(p_B)\right).$$

Using the universal property of (co)products one can show that

$$F((f,g)) \equiv (F(f), F(g)) \quad : \quad F(A) \longrightarrow F(B) \oplus F(B)$$

$$F\begin{pmatrix} 1_B \\ 1_B \end{pmatrix} = \begin{pmatrix} F(1_B) \\ F(1_B) \end{pmatrix} \quad : \quad F(B) \oplus F(B) \longrightarrow F(B)$$

Combining the two facts, one gets:

$$F(f+g) = F(f) + F(g)$$

Therefore that every left exact functor is additive..