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Higher étale cohomology groups of curves over $\bar{k}$

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We shall closely follow two chapters of the Stacks project [dJ+16], which can be found under 0A2M and 03RH.

In this talk X is an projective smooth curve of genus g over an algebraically closed field $k = \bar{k}$. Furthermore we assume $\mathrm{char}(k) \nmid n$.

Recollection

The Jacobian of the curve X is denoted by $\mathrm{Pic}^0(X)$ and the kernel of the multiplication-by-n map is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

We have the Kummer sequence $0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{()^n} \mathbb{G}_{m,X} \to 0$ and it's long exact sequence of cohomology

$$\begin{array}{l} 0 \longrightarrow \mu_n(k) \longrightarrow k^\times \xrightarrow{()^n} k^\times \xrightarrow{\delta} \\ H^1(X_{et}, \mu_n) \longrightarrow \mathrm{Pic}(X) \xrightarrow{()^n} \mathrm{Pic}(X) \xrightarrow{\delta} \\ H^2(X_{et}, \mu_n) \longrightarrow H^2(X_{et}, \mathbb{G}_m) \xrightarrow{()^n} H^2(X_{et}, \mathbb{G}_m) \xrightarrow{\delta} \\ H^3(X_{et}, \mu_n) \longrightarrow H^3(X_{et}, \mathbb{G}_m) \longrightarrow \ldots \end{array}$$

1 Fundamental sequence

Let $j : \mu \to X$ the inclusion of the generic point, $i_x : x \to X$ the inclusion of a closed point and X^0 the set of closed points.

Theorem 1. *There is a short exact sequence of sheaves on X_{et}*

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,\mu} \longrightarrow \bigoplus_{x \in X^0} i_{x*}\mathbb{Z} \longrightarrow 0.$$

We refer to this sequence as the fundamental sequence.

Proof. Let $U \to X$ an etale morphism and as U is the union of smooth, connected curves (see 03PC) we can assume U to be connected, hence irreducible. Then we have the exact sequence

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^\times) \longrightarrow k(U)^\times \xrightarrow{div} \bigoplus_{x \in U^0} i_{x*}\mathbb{Z},$$

which gives rise to the sequence

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^\times) \longrightarrow \Gamma(\mu \times_X U, \mathcal{O}_{\mu\times_X U}^\times) \longrightarrow \bigoplus_{x\in U^0} \Gamma(x \times_X U, \mathbb{Z})$$

and this is the same as

$$0 \longrightarrow \mathbb{G}_{m,X}(u) \longrightarrow j_*\mathbb{G}_{m,\mu}(U) \longrightarrow \left(\bigoplus_{x\in X^0} i_{x*}\mathbb{Z}\right)(U).$$

Only exactness at the last position remains, but recall that U is a nonsingular curve. Moreover, for a morphism of sheaves surjectivity and local surjectivity are equivalent. As a Zariski covering is also an etale covering, if a map is locally surjective for the Zariski topology it is also locally surjective for the etale topology. Nikitas showed the local surjectivity for the map $j_*\mathbb{G}_{m,\mu} \to \bigoplus_{x\in X^0} i_{x*}\mathbb{Z}$, so we obtain the result. □

We will use this Theorem to show that $H^p(X, \mathbb{G}_m) = 0$ for $p > 1$ by showing that the higher cohomology of $j_*\mathbb{G}_{m,\mu}$ and $i_{x*}\mathbb{Z}$ vanishes and for this result, we need some results from Galois cohomology.

2 Galois cohomology

Definition. Two finite central simple algebras A_1, A_2 over K are *similar* iff there are $m, n \geq 1$ such that $\mathrm{Mat}(n\times n, A_1) \cong \mathrm{Mat}(m\times m, A_2)$ as k-algebras.

Definition. The *Brauer Group* of a field K is the set $\mathrm{Br}(K)$ of similarity classes of finite central simple algebras over K.

One can read more about the Brauer groups in Chapter 11 of the Stack project (Tag 073W [dJ+16]) or section 50.60 (Tag 03R1 [dJ+16]), we will only use the fact that they exist.

Proposition 2. *Let K be a field with separable algebraic closure K^{sep}. Assume that for any finite extension K' of K we have* $\mathrm{Br}(K') = 0$. *Then*

1. $H^q(\mathrm{Gal}(K^{sep}/K), (K^{sep})^\times) = 0$ *for all* $q \geq 1$, *and*
2. $H^q(Gal(K^{sep}/K), M) = 0$ *for any torsion* $Gal(K^{sep}/K)$*-module* M *and any* $q \geq 2$.

For a proof see Chapter II, Section 3 of [Ser02].

Definition. A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, ..., T_n]_d$, there exist $a = (a_1, \cdots, a_n)$ with $a_i \in K$ not all zero, such that $f(a) = 0$.

Theorem 3. *If a field K is C_1 then* $\mathrm{Br}(K) = 0$.

Proof. We will use the fact that every element of $\mathrm{Br}(K)$ contains a unique central division K-algebra up to isomorphism. Let D be a finite dimensional division algebra over K with center K, then another fact about central simple algebras is that

$$D \otimes_K K^{sep} \cong \mathrm{Mat}_d(K^{sep}).$$

Moreover, the determinant $\det : \mathrm{Mat}_d(K^{sep}) \to K^{sep}$ descends to a homogeneous degree d map

$$\det : D \to K.$$

Now we use that K is C_1 and thus for $d > 1$ there is a nonzero $x \in D$ such that $\det(x) = 0$, but that would imply that x is not invertible, which is a contradiction. So D has degree 1, hence $\mathrm{Br}(K) = 0$. □

Theorem 4 (Tsens theorem)**.** *The function field of a variety of dimension r over an algebraically closed field k is C_r.*

For a proof see Tag 03RD [dJ+16].

Corollary 5. *Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero, that is* $\mathrm{Br}(k(C)) = 0$.

Out last Lemma of this section relies strongly on Corollary 5 and Proposition 2.

Lemma 6. *Let $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q(Spec(K)_{et}, \mathbb{G}_m) = 0$.*

Proof. There is the result $H^q(Spec(K)_{et}, \mathbb{G}_m) = H^q(Gal(K^{sep}/K), (K^{sep})^\times)$ and thus by Proposition 2 we only have to look at finite field extensions K' of K. Br preserves colimits and we can write K' as the colimit over L such that $k \subset L$ is finitely generated, of transcendence degree 1 and a subextension $L \subset K'$. But these L correspond to the function fields of curves over k and thus by Corollary 5 we have $\mathrm{Br}(K') = \mathrm{colim}\, 0$ □

3 Completing the sequences

Lemma 7. *For any $p \geq 1$ we have*

1. $R^p j_* \mathbb{G}_{m,\mu} = 0$,
2. $H^p(X_{et}, j_* \mathbb{G}_{m,\mu}) = 0$, *and*
3. $H^p(X_{et}, \bigoplus_{x \in X^0} i_{x*}\mathbb{Z}) = 0$

Proof. The proof of 1. uses Lemma 6 and for the details see Tag 03RJ [dJ+16]. For 2. we need to use the Leray spectral sequence

$$E_2^{p,q} = H^p(X_{et}, R^q j_* \mathbb{G}_{m,\mu}) \Rightarrow H^{p+q}(\mu, \mathbb{G}_{m,\mu}),$$

which vanishes for $p + q \geq 1$ by Lemma 6. The desired result is obtained using $q = 0$. For 3. we use that X is quasi-compact and quasi-separated and thus cohomology commutes with direct products. Now we can reduce to the case $H^p(X_{et}, i_{x*}\mathbb{Z}) = 0$ and note that i_x is finite, hence $R^q i_{x*}\mathbb{Z} = 0$ for $q > 0$ (Tag 03QP [dJ+16]). Again we have to use the Leray spectral sequence and get

$$H^p(X_{et}, i_{x*}\mathbb{Z}) = H^p(x_{et}, \mathbb{Z}).$$

Now recall that x is a closed point, in other words is the spectrum of an algebraically closed field. □

Corollary 8. *For $p \geq 2$ we have $H^p(X_{et}, \mathbb{G}_m) = 0$.*

Proof. Applying Lemma 7.2. and Lemma 7.3. to the long exact sequence of cohomology coming from the fundamental sequence. □

Lemma 9. *Let X be a smooth projective curve of genus g over k. Then there are canonical identifications*

$$H^q(X_{et}, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0, \\ \mathrm{Pic}^0(X)[n] & \text{if } q = 1, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. The results for $q \in \{0, 1\}$ we have see in previous talks. Furthermore using Corollary 8 we get the result for $q \geq 3$ immediately:

$$\begin{array}{l} 0 \longrightarrow \mu_n(k) \longrightarrow k^\times \xrightarrow{()^n} k^\times \xrightarrow{\delta} \\ \mathrm{Pic}^0(X)[n] \longrightarrow \mathrm{Pic}(X) \xrightarrow{()^n} \mathrm{Pic}(X) \xrightarrow{\delta_1} \\ H^2(X_{et}, \mu_n) \longrightarrow 0 \xrightarrow{()^n} 0 \xrightarrow{\delta} \\ H^3(X_{et}, \mu_n) \longrightarrow 0 \longrightarrow \ldots \end{array}$$

We know that $H^2(X_{et}, \mu_n) \cong \operatorname{coker} \delta_1$ and then consider the commutative diagram with exact rows and columns and apply the snake lemma

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X)[n] & \xrightarrow{\cong} & \ker{()^n} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0\\
 & & {\scriptstyle ()^n}\downarrow & & {\scriptstyle ()^n}\downarrow & {\scriptstyle \delta} & {\scriptstyle n}\downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \operatorname{coker}{()^n} & \xrightarrow{\cong} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

□

References

[dJ+16] Aise Johan de Jong et al. Stacks project. open source project, 2016.

[Ser02] Jean-Pierre Serre. Galois cohomology, corrected reprint of the 1997 english edition. *Springer Monographs in Mathematics, Springer-Verlag, Berlin*, pages 94720–3840, 2002.