H^1 and torsors

Luca Giovenzana

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1 Locally isomorphic sheaves

Sheaves on a topological space (or more generally on a site) are objects that allow us to speak of how you can glue things locally to define something globally. Then, it is almost a tautology that in order to define a morphism between sheaves it is enough to define a family of morphisms on a covering in such a way they are compatible. In other words we have:

Proposition 1.1. *Let $\mathcal{C}$ be a site and $\mathcal{X}, \mathcal{Y} \in Sh(\mathcal{C})$ be two sheaves of sets (or groups, rings, etc.). Then the two presheaves $\mathcal{H}om(\mathcal{X}, \mathcal{Y})$, $\mathcal{I}som(\mathcal{X}, \mathcal{Y})$, defined by:*

$$\begin{aligned} \mathcal{H}om(\mathcal{X}, \mathcal{Y}) : & \mathcal{C}^{op} \to Set \\ & X \mapsto \mathrm{Hom}_{Sh(\mathcal{C}/X)}(\mathcal{X}_{|X}, \mathcal{Y}_{|X}) \end{aligned}$$

$$\begin{aligned} \mathcal{I}som(\mathcal{X}, \mathcal{Y}) : & \mathcal{C}^{op} \to Set \\ & X \mapsto \mathrm{Isom}_{Sh(\mathcal{C}/X)}(\mathcal{X}_{|X}, \mathcal{Y}_{|X}) \end{aligned}$$

(where the associations on the arrows are understood) are sheaves.

Definition 1.1. Two sheaves $\mathcal{X}, \mathcal{Y}$ over a site $\mathcal{C}$ are said to be locally isomorphic if for every object X of $\mathcal{C}$ there exists a covering $\{U_i \to X\}_{i \in I}$ such that $\forall i \in I$ $\mathcal{X}_{|U_i} \cong \mathcal{Y}_{|U_i}$.

Example 1.1. Let A be an algebra over a field k with unity, associative, not necessarily commutative, finitely generated as k-module. A is said to be central simple if $k = Z(A)$ (the center of A) and it doesn't have two-sided ideals, beside $\{0\}$ and A.
There is a canonical sheaf associated to the algebra A on $(\mathrm{Spec}(k))_{\acute{e}t}$, namely the unique one such that $\widehat{A}(L) := L \otimes_k A$ for every finite separable field extension $k \subseteq L$. Since every central simple algebra is split by a finite separable extension, the sheaves associated to algebras of same degree are locally isomorphic.
If k is a field, the quaternion algebra associated to k is given by the algebra with basis $\{1, i, j, k\}$ where $i^2 = -1, j^2 = -1, k = ij = -ji$. It is a central simple algebra, moreover it is a division algebra.

Example 1.2. Consider $(X, \mathcal{O})$ a scheme, then locally free sheaves of rank n are locally isomorphic, indeed by definition they are defined as sheaves locally isomorphic to $\mathcal{O}^n$. This example can be generalized to locally free sheaves over a ringed site where a ringed site is a pair $(\mathcal{C}, \mathcal{O})$ with $\mathcal{C}$ a site and $\mathcal{O}$ a sheaf of rings over it.

Exercise 1.1. *let $Y \to X$ a finite étale cover, then it is locally trivial.*

Hint: proceed by induction on the degree of the covering. Consider the base change along the same map f; you can consider the section given by the diagonal map $\Delta : Y \to Y \times_X Y$, it is an open immersion then...

Torsors

Let $\mathcal{G}$ be a sheaf of groups on a site $\mathcal{C}$; a left-action of $\mathcal{G}$ on a sheaf of sets $\mathcal{X}$ is a natural transformation, say $\alpha : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$, s.t. for every object U in $\mathcal{C}$, $\alpha(U)$ defines an action of $\mathcal{G}(U)$ on

$\mathcal{X}(U)$. Consider the category of sheaves of sets with a $\mathcal{G}$-action: its objects are $(\mathcal{X}, \alpha)$ with $\mathcal{X}$ and α as above; a morphism between two objects $(\mathcal{X}, \alpha)$ and $(\mathcal{Y}, \beta)$ is given by a morphism of sheaves $f : \mathcal{X} \to \mathcal{Y}$ compatible with the $\mathcal{G}$-action; in other words $\forall U$ in $\mathcal{C}$ there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}(U) \times \mathcal{X}(U) & \xrightarrow{\alpha_U} & \mathcal{X}(U) \\ {\scriptstyle id(U)\times f(U)}\downarrow & & \downarrow{\scriptstyle f(U)} \\ \mathcal{G}(U) \times \mathcal{Y}(U) & \xrightarrow{\beta_U} & \mathcal{Y}(U) \end{array}$$

Similarly one can define sheaves of sets with a right-action for a sheaf of group; from now on, if not stated differently, by torsors we will mean left-torsors and by action left-action.

Definition 1.2. Let $(\mathcal{X}, \alpha)$ the pair given by a sheaf of sets with a $\mathcal{G}$-action α, then $(\mathcal{X}, \alpha)$ is said to be a $\mathcal{G}$-torsor if for every object X in $\mathcal{C}$, there exists a covering $\{U_i \to \mathcal{X}\}_{i \in I}$ such that $\forall i \in I$ $(\mathcal{X}_{|U_i}, \alpha_i)$ is isomorphic to $(\mathcal{G}_{|U_i}, m_{|U_i})$, where $m_{|U_i}$ denotes the (left-)action given by the multiplication.

Equivalently, a sheaf of sets $\mathcal{X}$ with a $\mathcal{G}$-action is a $\mathcal{G}$-torsor if for every object X in $\mathcal{C}$ the action of $\mathcal{G}(X)$ on $\mathcal{X}(X)$ is free and transitive and there exists a covering $\{U_i \to X\}_{i \in I}$ such that for every $i \in I$ $\mathcal{X}(U_i) \neq \emptyset$.

Remark 1.1.

- Notice that a $\mathcal{G}$-torsor $\mathcal{X}$ on a topological space X is trivial if and only if $\mathcal{X}(X) \neq \emptyset$.
- the category of $\mathcal{G}$-torsors is a groupoid: every morphism is an isomorphism.

Example 1.3. If $\mathcal{X}$ and $\mathcal{Y}$ are locally isomorphic sheaves, then $\mathcal{I}som(\mathcal{X}, \mathcal{Y})$ is naturally a bi-torsor with the sheaves of groups $\mathrm{Aut}(\mathcal{X})$, $\mathrm{Aut}(\mathcal{Y})$ acting on the right and left respectively.

Definition 1.3. Given two sheaves of sets on a site $\mathcal{C}$, $\mathcal{X}$ with a right $\mathcal{G}$-action and $\mathcal{Y}$ with a left $\mathcal{G}-$action, we can consider the (left-)action of $\mathcal{G}$ on the product given for every object X by:

$$\begin{aligned} (\mathcal{G} \times \mathcal{X} \times \mathcal{Y})(X) &\to (\mathcal{X} \times \mathcal{Y})(X) \\ (g, x, y) &\mapsto (xg^{-1}, gy) \end{aligned}$$

Then we define the sheaf contracted product of $\mathcal{X}$ and $\mathcal{Y}$, denoted $\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$ (or in the literature $\mathcal{X} \wedge^{\mathcal{G}} \mathcal{Y}$), to be the sheafification of the presheaf defined as the quotient by the action of $\mathcal{G}$:

$$\begin{aligned} &\mathcal{C}^{op} \to Sets \\ &X \mapsto \mathcal{X}(X) \times \mathcal{Y}(X)/\sim \end{aligned}$$

i.e. the quotient by the equivalence relation defined by $(xg, y) \sim (x, gy)$ $\forall g \in \mathcal{G}(X)$.

Remark 1.2.

- If we fix the sheaf $\mathcal{X}$, then $\mathcal{X} \otimes_{\mathcal{G}} \_$ defines a functor.
- the contracted product can be characterized by meaning of a universal property; namely, $\forall f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ morphism of sheaves, s.t. for every object U $\forall g \in \mathcal{G}(U)$ $f(U)(xg, y) = f(U)(x, gy)$ then there exists a unique morphism $\varphi : \mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y} \to \mathcal{Z}$ s.t. $\varphi \circ p = f$, where $p : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$ is the quotient map.

Example 1.4.

- Let $(X, \mathcal{O}_X)$ a ringed site, then the functor:

  $$\begin{gathered} \{GL_{n,X}\text{-torsors}\} \to \{\text{locally free } \mathcal{O}_X\text{-modules of rank n}\} \\ \mathcal{T} \mapsto \mathcal{T} \otimes_{GL_{n,X}} \mathcal{O}_X^n \end{gathered}$$

  defines an equivalence of categories with quasi-inverse given by:

  $$\mathcal{E} \mapsto \mathcal{I}som(\mathcal{O}_X^n, \mathcal{E})$$

  (where the associations on the arrows are the obvious ones.)
  To see this, notice that the morphism of sheaves f,defined for every U, by

  $$\begin{aligned} f(U) : \mathcal{I}som(\mathcal{O}^n, \mathcal{E})(U) \times \mathcal{O}^n(U) &\to \mathcal{E}(U) \\ (\varphi, s) &\mapsto \varphi(U)(s) \end{aligned}$$

  factors through the contracted product and it is easy to check that locally it defines an isomorphism.

- If $f : \mathcal{H} \to \mathcal{G}$ is a morphism of sheaves of groups, then $\mathcal{H}$ acts on $\mathcal{G}$ and the contracted product defines a functor between the categories of torsors:

$$\begin{aligned} f^* : \{\mathcal{H}\text{-torsors}\} &\to \{\mathcal{G}\text{-torsors}\} \\ \mathcal{T} &\mapsto \mathcal{T} \otimes_{\mathcal{H}} \mathcal{G} \end{aligned}$$

The first cohomology group:

The language of torsors allows us to give a description of the first cohomology set:

Theorem 1.1. *Let $(\mathcal{C}, \mathcal{O})$ be the ringed site with $\mathcal{C}$ $X_{\acute{e}t}$ or X_{Zar} with the relative sheaf of rings; let $\mathcal{F}$ be an $\mathcal{O}$-module, then there is a natural bijection:*

$$H^1(\mathcal{C}, \mathcal{F}) \simeq \{\mathcal{F}\text{-}\textit{torsors}\}/\sim$$

Proof. (sketch)
I will split the proof in two steps: in the first one I'll show that $H^1(\mathcal{C}, \mathcal{F}) \simeq \mathrm{Ext}^1(\mathcal{O}, \mathcal{F})$ and in the second one I'll show a bijection between $\mathrm{Ext}^1(\mathcal{O}, \mathcal{F})$ and $\{\mathcal{F}\text{-torsors}\}/\sim$.
Recall that $Ext^1(\mathcal{O}, \mathcal{F})$ is defined to be the group of extensions of $\mathcal{O}$ by $\mathcal{F}$, i.e. $\mathrm{Ext}^1(\mathcal{O}, \mathcal{F}) = \{0 \to \mathcal{F} \to E \to \mathcal{O} \to 0\}/\sim$, where the law group is defined by the Baer sum and the neutral element is given by the split sequence. In order to prove the first step, recall consider an the exact sequence:

$$0 \to \mathcal{F} \to I \to Q \to 0$$

where I is injective. Since the functor $\Gamma(\mathcal{C}, \_) : \mathrm{Sh}(\mathcal{C})$ can be rewritten as $\mathrm{Hom}(\mathcal{O}, \_)$ proving the claim is equivalent to show that:

$$0 \to \mathrm{Hom}(\mathcal{O}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}, I) \xrightarrow{\varphi} \mathrm{Hom}(\mathcal{O}, Q) \xrightarrow{\Psi} \mathrm{Ext}^1(\mathcal{O}, \mathcal{F}) \to 0$$

is exact, i.e. that there exists a morphism Ψ such that the pair $(\mathrm{Ext}^1(\mathcal{O}, \mathcal{F}), \Psi) \simeq$cokernel$(\varphi)$. For a morphism $\mathcal{O} \to Q$ define $\Psi(f)$ to be the exact sequence obtained by the pull-back along f:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & P & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & I & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

It is easy to see that $\Psi(f)$ is a split exact sequence if and only if f factorizes through I, i.e. it lies in the image of φ. Moreover Ψ can be checked to be surjective. This concludes the first step.

In order to show the second step, I will define $\alpha : \mathrm{Ext}^1(\mathcal{O}, \mathcal{F}) \to \{\mathcal{F}\text{-torsors}\}/\simeq$ and its inverse. Consider an exact sequence $s : \ o \to \mathcal{F} \to E \to \mathcal{O} \to 0$, then since it is locally split, E is locally isomorphic to $\mathcal{F} \oplus \mathcal{O}$; let's define $\alpha(s) = \mathcal{I}som_{\mathrm{Ext}}(\mathcal{F} \oplus \mathcal{O}, E)$ where the action of $\mathcal{F}$ is induced by the morphism $\mathcal{F} \to \mathrm{Aut}(\mathcal{F} \oplus \mathcal{O})$, which associates to every element f the morphism represented by the matrix:

$$\begin{pmatrix} id & f \\ 0 & id \end{pmatrix}$$

To define $\beta : \{\mathcal{F}\text{-torsors}\}/\simeq \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{F})$, the inverse of α, consider an $\mathcal{F}$-torsor $\mathcal{T}$ and the split exact sequence $0 \to \mathcal{F} \to \mathcal{F} \oplus \mathcal{O} \to \mathcal{O} \to 0$, then let define $\beta(\mathcal{T})$ to be the exact sequence:

$$0 \to \mathcal{T} \otimes_{\mathcal{F}} \mathcal{F} \to \mathcal{T} \otimes_{\mathcal{F}} (\mathcal{F} \oplus \mathcal{O}) \to \mathcal{F} \otimes_{\mathcal{F}} \mathcal{O} \to 0$$

where the action on $\mathcal{F} \oplus \mathcal{O}$ is the same as above and the one on $\mathcal{F}$ and $\mathcal{O}$ is the trivial one. α and β so defined are inverse each other. ☐

Remark 1.3. The map f^* described in the previous example induces the map between the cohomology sets $H^1(\mathcal{C}, H) \to H^1(\mathcal{C}, \mathcal{G})$

In general, for a sheaf of (not necessarily commutative) groups $\mathcal{G}$, the first cohomology group $H^1(\mathcal{C}, \mathcal{G})$ is a pointed set, it has the distinguished element given by the isomorphism class of the trivial torsor $(\mathcal{G}, m)$.

We conclude with an example of a non-trivial twist of $\mathbb{G}_a$.

Example 1.5. Let k be a field of characteristic 2 and let $a \in k$ an element that is not a square; then the polynomial $f(x) = y^2z - x^3 - ax^2z$ defines a cubic curve $E_a \subseteq \mathbb{P}^2_k$. Notice that it is a singular curve, indeed it has a cusp at the point $(0 : 0 : 1)$. The non-singular points of E_a^{ns} are isomorphic to $\mathbb{G}_a$, but only after a base change to a k-algebra containing a square root of a, in particular the isomorphism is not realized over any separable extension of k. To see this last statement, notice that a non-trivial twist of $\mathbb{G}_a$ over $\mathrm{Spec}(k)_{\acute{e}t}$ would give a non-trivial torsor of $\mathrm{Aut}(\mathbb{G}_a) \simeq \mathbb{G}_m$ (as sheafs over $\mathrm{Spec}(k)_{\acute{e}t}$), but it's known by Hilbert's theorem 90 that $\mathrm{H}^1(\mathbb{G}_m) = 0$.

References:

I don't claim any originality for these notes, you can find everything in greater detail in:

[1] http://pub.math.leidenuniv.nl/~jinj/2014/ec/etale3.pdf
[2] https://www.math.leidenuniv.nl/scripties/MasterZomervrucht.pdf
[3] http://pub.math.leidenuniv.nl/~edixhovensj/talks/2011/2011_03_15.pdf