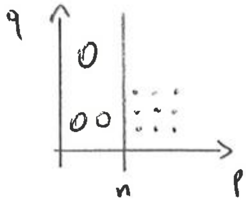
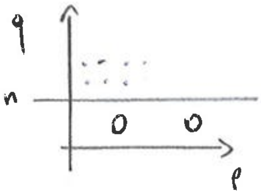
Luca Giovenzana - 11.5.16

Review of the main things

- From a double complex $C^{\bullet,\bullet}$ one can get 2 filtrations of $\mathrm{Tot}(C^{\bullet,\bullet})$





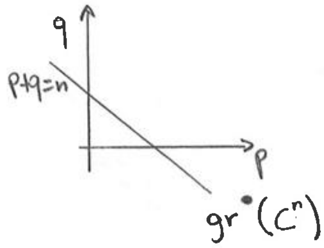
If $C^{\bullet,\bullet}$ is 1^{st} quadrant
the filtrations are canonically bounded, i.e.

- $F^0(\mathrm{Tot}(C^{\bullet,\bullet})^k) = \mathrm{Tot}(C^{\bullet,\bullet})^k \quad \forall k$
- $F^{n+1}(\mathrm{Tot}(C^{\bullet,\bullet})^n) = 0$

$$F^n(\mathrm{Tot}(C^{\bullet,\bullet})^k) = \bigoplus_{\substack{p+q=k \\ p\geq n}} C^{p,q}$$

- From a canonically bounded filtration of a complex one can get a 1^{st} quadrant spectral sequence converging to the cohomology of the complex

$\boxed{r=0}$



$$E_0^{p,q} = gr^p(C^{p+q}) = \frac{F^p(C^\bullet)}{F^{p+1}(C^\bullet)}$$

- Theo:

Combining the 2 previous points one gets 2 S.S. converging to the cohomology of the total complex

$$'E_2^{p,q} = H^p_{hor}(H^q_{ver}(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$$
$$''E_2^{p,q} = H^p_{ver}(H^q_{hor}(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet})^\bullet)$$

Theo - Grothendieck spectral sequence

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories.
Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives
Left $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$ be left exact (additive) functors
Assume that $\forall I \in \mathcal{A}$ injective object $G(I) \in \mathcal{B}$ is F-acyclic

Then
$\exists$ a 1^{st} quad. S.S. $\forall A \in \mathcal{A}$
$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow (R^{p+q}(FG))(A)$$

Lemma

Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact (additive) functor between AB categories
Assume that $\mathcal{A}$ has enough injectives
Then $\forall A^\bullet \in Ch^+(\mathcal{A})$ there are S.S.:

$$'E_2^{p,q} = H^p(R^qF(A^\bullet)) \Rightarrow R^{p+q}F(A^\bullet) \qquad ''E_2^{p,q} = (R^pF)(H^q(A^\bullet)) \Rightarrow R^{p+q}F(A^\bullet)$$

Proof (LEMMA):

LET ~~[illegible]~~ $A^\bullet \to I^{\bullet,\bullet}$ BE A C.E. RESOLUTION OF ~~[illegible]~~ $A^\bullet$; $F(I^{\bullet,\bullet})$ IS A DOUBLE COMPLEX OF $\mathcal{B}$

$$'E_2^{p,q} = H^p_{hor} \underbrace{H^q_v\left(F(I^{\bullet,\bullet})\right)}_{= R^qF(A^\bullet)} \Rightarrow H^{p+q}\left(\mathrm{Tot}\, F(I^{\bullet,\bullet})\right) \overset{\text{by def.}}{=} \mathbb{R}^{p+q} F \, (A^\bullet)$$

$$\text{so} \quad 'E_2^{p,q} = H^p_h\left(R^qF\,(A^\bullet)\right)$$

ON THE OTHER HAND,

$$''E_2^{p,q} = H^p_v H^q_h\left(F(I^{\bullet,\bullet})\right) \overset{\star}{=} R^pF\left(H^q(A^\bullet)\right)$$

$\star$ $I^{\bullet,q}$ IS AN INJECTIVE ~~[illegible]~~ COMPLEX WITH INJECTIVE BOUNDARIES B^p_h, SO ~~$H^q_h(F(I^{\bullet,\bullet})) = H^q(A^\bullet)$~~

$$H^p_v H^q_h\left(F(I^{\bullet,q})\right) = H^p_v\left(F\left(H^q_h(I^{\bullet,\bullet})\right)\right)$$

~~$H^q(I^{\bullet,\bullet})$ [illegible] INJECTIVE~~

$H^q_h(I^{\bullet,\bullet})$ ARE INJECTIVE RESOLUTION (MAKING q VARY) SO $H^p\left(F\left(H^q_h(I^{\bullet,\bullet})\right)\right) = R^pF\left(H^q_h(I^{\bullet,\bullet})\right)$

HENCE THE THESIS $\square$

Proof (THEO):

LET $A \to I^\bullet$ BE AN INJECTIVE RES. $\quad G(I^\bullet) \in Ch^+(\mathcal{B})$

NOW USE THE LEMMA (WITH $A^\bullet$ OF THE LEMMA DEFINED AS $A^\bullet := G(I^\bullet)$):

$$'E_2^{p,q} = H^p\left(R^qF\left(G(I^\bullet)\right)\right) \Rightarrow \left(\mathbb{R}^{p+q}F\right)\left(G I^\bullet\right)$$

G SENDS INJECTIVE IN F-ACYCLIC, SO $(R^qF)\left(G(I^p)\right) = 0 \quad \forall q \neq 0 \ \forall p$

$$\text{SO} \quad 'E_2^{p,q} = H^p\left(R^qF\left(G(I^\bullet)\right)\right) = H^p\left(FG(I^\bullet)\right) = \left(R^p(FG)\right)(A) \qquad 1 \text{ ROW } (q=0)$$

$$\text{THUS} \quad ''E_2^{p,q} = (R^pF)\, H^q\left(G(I^\bullet)\right) \Rightarrow R^p(FG)(A)$$

BUT $H^q\left(G(I^\bullet)\right) = R^qG(A)$ I.E. $R^pF\left(R^qG(A)\right)$ $\square$

LERAY S.S.

LET $f : X \to Y$ BE A CONTINUOUS MAP OF TOPOLOGICAL SPACES (+ étale site), WE CAN CONSIDER THE FUNCTORS:

$$Ab(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Ab(Y)$$

~~$f^{-1} \dashv f_*$ AND f^{-1} IS EXACT THUS f_* IS LEFT EXACT AND PRESERVES INJECTIVES~~

THEN $\forall \mathcal{F} \in Ab(X)$ THERE EXISTS A S.S.:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Proof:

- $f^{-1} \dashv f_*$ AND f^{-1} EXACT
  THUS f_* IS LEFT EXACT AND PRESERVES INJECTIVES
- $f_* \mathcal{F}(Y) \overset{\text{by def}}{=} \mathcal{F}(f^{-1}\cancel{Y}) = \mathcal{F}(X)$ I.E.

$$\begin{tikzcd} Ab(X) \arrow[r, "f_*"] \arrow[dr, "\Gamma"'] & Ab(Y) \arrow[d, "\Gamma"] \\ & Ab \end{tikzcd}$$

COMMUTES

APPLY GROTHENDIECK THEOREM AND GET THE THESIS $\square$

RMK

IF $f : X \to Y$ MORPHISM OF SCHEMES

THE SAME PROOF FOR

$$\begin{tikzcd} Ab(X_{\acute{e}t}) \arrow[r, "f_*"] \arrow[dr, "\Gamma"'] & Ab(Y) \arrow[d, "\Gamma"] \\ & Ab \end{tikzcd}$$

SHOWS THAT $\forall \mathcal{F} \in Ab(X)$ WE HAVE:

$$E_2^{p,q} = H^p(Y_{\acute{e}t}, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Prop. III.8.1 Hartshorne

Let $f: X \to Y$ be ~~a map of topological sp~~ a continuous map between topological spaces

$f_*: Ab(X) \to Ab(Y)$ is left exact

$\forall q \geq 0$ one has that $\forall \mathcal{F} \in Ab(X)$

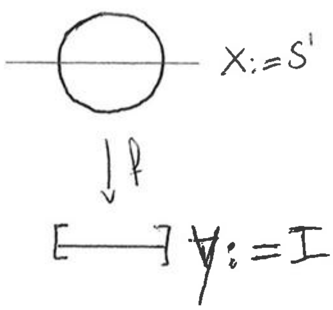
$R^q f_* \mathcal{F}$ is the sheaf associated to the presheaf defined by the association:

$$\mathrm{Open}(Y) \longrightarrow \mathbf{Ab}$$
$$V \longmapsto H^i\left(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}\right)$$

This tool will allow us to give an example.

Example

Consider the projection of S^1 onto the unit interval I.



We want to compute the cohomology of the constant sheaf $\mathbb{Q}_{S^1} \in Ab(S^1)$

In order to use the Leray spectral sequence we need to compute the cohomology of the functors $R^q f_* \mathbb{Q}_{S^1} \in Ab(I)$ $q \geq 0$

Thanks to the previous proposition it's easily seen that $(R^q f_*)\mathbb{Q} = 0$ if $q > 0$.

While for $q = 1$ one sees that the stalks are given by:

$$(f_* \mathbb{Q})_x = \begin{cases} \mathbb{Q}^2 & \text{if } x \in (0,1) \\ \mathbb{Q} & \text{if } x \in \{0,1\} \end{cases}$$

Moreover one notices that $\mathbb{Z}/2\mathbb{Z}$ acts on X by a symmetry

i.e. there is a morphism of groups $\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Aut}_Y(X)$.

This gives me a decomposition

$$f_* \mathbb{Q} = (f_* \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}} \oplus (f_* \mathbb{Q})^-$$ where $(f_* \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$ denotes the invariants for the symmetry.

It is clear that $(f_* \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}_I$ the constant sheaf

$(f_* \mathbb{Q})^- = j_! \mathbb{Q}_{(0,1)}$ where j is the inclusion $(0,1) \hookrightarrow [0,1]$

To compute the cohomology of $f_* \mathbb{Q}$ consider the S.E.S:

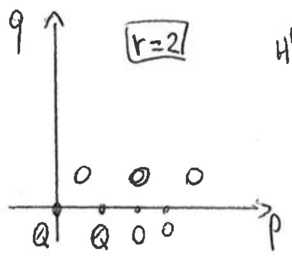
$$0 \to j_! \mathbb{Q}_{(0,1)} \to \mathbb{Q}_{[0,1]} \to i_{0,*}\mathbb{Q} \oplus i_{1,*}\mathbb{Q} \to 0$$

Taking the long exact sequence in cohom. we get

$$\{0\} \hookrightarrow \mathbb{Q} \xrightarrow{\Delta} \mathbb{Q} \oplus \mathbb{Q}$$
$$\hookrightarrow H^1(I, j_! \mathbb{Q}_{(0,1)}) \to 0 \to 0$$

Thus we get $H^1(I, j_! \mathbb{Q}_{(0,1)}) \cong \mathbb{Q}$

The Leray spectral sequence looks like



$$H^p(I, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(S^1, \mathbb{Q})$$

Hence we get

$H^0(S^1, \mathbb{Q}) \cong \mathbb{Q}$

$H^1(S^1, \mathbb{Q}) \cong \mathbb{Q}$

as expected.

WE SAY THAT AN ABELIAN GROUP M IS A G-MODULE IF IT HAS A G-ACTION;
THEN WE HAVE AN ISOMORPHISM OF CATEGORIES $\underline{G\text{-Mod}} \xrightarrow{\sim} \mathbb{Z}[G]\text{-Mod}$ WHERE $\mathbb{Z}[G]$ IS THE GROUP ALGEBRA

WE CAN THEN CONSIDER THE FOLLOWING ~~CATEGOR~~ FUNCTORS:

$$\begin{tikzcd} \underline{\mathbb{Z}[G]\text{-Mod}} \arrow[r, shift left=3, "(-)^G"] \arrow[r, shift right=3, "(-)_G"'] & \underline{Ab} \arrow[l, "T"'] \end{tikzcd}$$

~~WHERE T ASSOCIATES TO AN ABELIAN GROUP~~

WHERE

- $\forall A \in \underline{Ab}$ $T(A)$ IS THE G-MODULE WITH TRIVIAL ACTION
- $\forall M \in \mathbb{Z}[G]\text{-Mod}$ M^G IS THE ABELIAN GROUPS OF ELEMENTS FIXED BY G, I.E. $M^G := \{ m \in M \mid gm = m \ \forall g \in G \}$
- $\forall M \in \mathbb{Z}[G]\text{-Mod}$ M_G IS THE COINVARIANT OF M, I.E. $M_G := M / \langle gm - m \mid g \in G, m \in M \rangle$

NOTICE THAT THERE NATURAL ISOMORPHISMS OF FUNCTORS:

- $(-)^G \cong \mathrm{Hom}_G(T(\mathbb{Z}), -)$
- $(-)_G \cong T(\mathbb{Z}) \otimes_{\mathbb{Z}[G]} -$

IN PARTICULAR

T IS LEFT ADJOINT TO $(-)^G$ $\qquad T \dashv (-)^G$

AND $(-)_G$ IS LEFT ADJOINT TO T $\qquad (-)_G \dashv T$

THEN WE CAN CONSIDER THE DERIVED FUNCTORS

1. $H^i(G, A): \mathbb{Z}[G]\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}$
   $A \longmapsto R^i(-)^G(A)$

2. $H_i(G, A): \mathbb{Z}[G]\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}$
   $A \longmapsto (L_i(-)_G)(A)$

HOCHSCHILD-SERRE SPECTRAL SEQUENCE

LET $H \trianglelefteq G$ BE A NORMAL SUBGROUP, THEN THERE ARE S.S.:

$$E_2^{p,q} = H^p(G/H; H^q(H, A)) \Longrightarrow H^{p+q}(G, A)$$

$$E^2_{p,q} = H_p(G/H; H_q(H, A)) \Longrightarrow H_{p+q}(G, A) \qquad \forall A \in \underline{G\text{-Mod}}$$

proof:

CONSIDER THE FUNCTORS

$$\begin{tikzcd} G\text{-Mod} \arrow[r, "(-)_H"] \arrow[dr, "(-)_G"'] & G/H\text{-Mod} \arrow[d, "(-)_{G/H}"] \\ & Ab \end{tikzcd} \qquad \begin{tikzcd} G\text{-Mod} \arrow[r, "(-)^H"] \arrow[dr, "(-)^G"'] & G/H\text{-Mod} \arrow[d, "(-)^{G/H}"] \\ & Ab \end{tikzcd}$$

A G/H-MODULE IS JUST A G-MODULE ON WHICH THE ACTION OF H IS TRIVIAL, SO $\forall A \in G\text{-Mod}$ A^H AND A^H ARE G/H-MODULES

AS BEFORE

$$\begin{tikzcd} G\text{-Mod} \arrow[r, bend left, "(-)_H"] \arrow[r, bend right, "(-)^H"'] & G/H\text{-Mod} \arrow[l, "\pi^*"'] \end{tikzcd}$$

WHERE π^* IS THE FUNCTOR WHICH GIVES TO A G/H-MODULE THE G-ACTION INDUCED BY $G \xrightarrow{\pi} G/H$.

$$\begin{array}{l} (-)_H \dashv F \\ F \dashv (-)^H \end{array} \quad \text{AND } F \text{ IS EXACT SO} \quad \begin{array}{lll} (-)_H & \text{PRESERVES} & \text{PROJ.} \\ (-)^H & \text{"} & \text{INJ.} \end{array}$$

APPLY GROTHENDIECK S.S. AND GET THE THESIS. $\blacksquare$