Sheafification and existence of injectives

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1 Sheafification

In this section, we will discuss how a sheafification works on ANY small site. The presheaves we will discuss are presheaves of sets. But once this construction is done, one can immediately guess how this would work on presheaves of (abelian) groups and of (commutative) rings. The sheafification will be done in two steps: first we will turn presheaves into *separated* presheaves, and then we will turn separated presheaves into (actual) sheaves.

1.1 Separated sheaves

Definition 1.1. Let $\mathcal{C}$ be a site. A presheaf on $\mathcal{F}$ on $\mathcal{C}$ is called *separated* if for any object $U \in \mathcal{C}$, and any covering $(U_i)_i$, the restriction map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ is injective.

Definition 1.2. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. The *separated presheaf associated to* $\mathcal{F}$ is a separated presheaf $\mathcal{F}^s$, equipped with a morphism $\mathcal{F} \to \mathcal{F}^s$ satisfying the following universal property: for each morphism $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ being separated, there exists a *unique* $\mathcal{F}^s \to \mathcal{G}$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^s \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

Theorem 1.3. *The separated presheaf associated to $\mathcal{F}$ exists, and is given by $\mathcal{F}^s(U) = \mathcal{F}(U)/\sim$, with $s, t \in \mathcal{F}(U)$ being equivalent if there exists a covering $(U_i)_i$ such that $s|_{U_i} = t|_{U_i}$ for each i.*

Proof. See Exercise 3.1. □

1.2 Sheafification of separated sheaves

From now on, we will assume $\mathcal{C}$ to be a *small* site.

Definition 1.4. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. The *sheafification of* $\mathcal{F}$ is a sheaf $\mathcal{F}^+$, equipped with a morphism $\mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property: for each morphism $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ being a sheaf, there exists a unique $\mathcal{F}^+ \to \mathcal{G}$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

Theorem 1.5. *Let $\mathcal{F}$ be a separated presheaf. Then $\mathcal{F}$ has a sheafification given by:*

$$\mathcal{F}^+(U) = \{(s_i, U_i)_i : s_i \in \mathcal{F}(U_i); (U_i)_i \text{ covers } U; s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}} \text{ for all } i, i'\}/\sim$$

Here we have that $(s_i, U_i)_i$ and $(t_j, V_j)_j$ are equivalent if $s_i|_{U_i \times_U V_j} = t_j|_{U_i \times_U V_j}$ for all i, j.

Combining Theorem 1.3 and Theorem 1.5 gives us the following result:

Corollary 1.6. *Let $\mathcal{F}$ be any presheaf on $\mathcal{C}$. Then $\mathcal{F}$ has a sheafification $\mathcal{F} \to \mathcal{F}^+$.*

We know that sheafification preserves *colimits*; this is because because the sheafification functor is a *left adjoint functor*. However, does sheafification also preserve *limits*? Not quite. However, sheafification does preserve limits of *finite* diagrams. To prove this, it suffices to prove that sheafification preserves equalizers and finite products. This will be proven in Exercise 3.2.

2 Existence of injectives

We first need to know in which abelian category we will construct those injectives.

Definition 2.1. Let $\mathcal{C}$ be a *ringed site*, i.e. a a site equipped with a sheaf of rings. A sheaf of abelian groups $\mathcal{F}$ on $\mathcal{C}$ is called a $\mathcal{C}$*-module* if for each object, $\mathcal{F}(U)$ has the structure of a $\mathcal{C}(U)$-module, and the structure is compatible with restrictions. We call this category $\mathbf{Mod}_{\mathcal{C}}$.
A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups is called a $\mathcal{C}$*-module morphism* if on each U, $\mathcal{F}(U) \to \mathcal{G}(U)$ is also a $\mathcal{C}(U)$-module morphism.

Remark 2.2. If we equip $\mathcal{C}$ with the sheafification of the constant presheaf $\mathbb{Z}$, then we will obtain the category $\mathbf{Ab}_{\mathcal{C}}$ of all sheaves of abelian groups on $\mathcal{C}$.

While it is true that above category has enough injectives for any site, we will only concider a limited class of sites, namely the *small Étale sites*: sites in the form X_{et}, with X a scheme. On such site, we have the notion of a *stalk*, which allows us to mimic the case of topological spaces almost completely.

2.1 Pushforward and inverse image sheafs

In this section, we generalize the notion of inverse immage sheafs and pushforwards sheafs. This construction works for any functor of sites that is *continuous*, but for simplicity, we shall limit ourselves to morphisms of schemes. We shall start with the definition for pushforward sheafs:

Definition 2.3. Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a sheaf on X_{et}. Then the *pusforward sheaf* $f_*(\mathcal{F})$ on Y_{et} is given by:

$$f_*(\mathcal{F})(U) = \mathcal{F}(U \times_Y X)$$

The case for inverse image sheafs is more complicated, but still somewhat familiar to the case of topological spaces.

Definition 2.4. Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a sheaf on Y_{et}. Fix an Étale morphism $U \to X$, and consider the diagram of pairings (V, Φ), where $V \to Y$ is an Étale morphism, and $\Phi : U \to V$ is a morphism such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

The arrows in the the diagram $(V, \Phi) \to (W, \Psi)$ are morphisms $V \to W$ that are compatible with Φ and Ψ.
On this diagram, we can apply $\mathcal{F}$ by putting $\mathcal{F}(V, \Phi) = \mathcal{F}(V)$. Let $p(U)$ be the colimit of this diagram. This gives a presheaf p. The *inverse image sheaf* $f^{-1}(\mathcal{F})$ is the sheaf associated to the presheaf.

Much like with topological spaces, we get the following identification.

Lemma 2.5. *Let $f : X \to Y$ be a morphism of schemes. There exists a functorial isomorphism:*

$$\mathit{Hom}\ (f^{-1}(\mathcal{F}), \mathcal{G}) \cong \mathit{Hom}\ (\mathcal{F}, f_*(\mathcal{G}))$$

2.2 Stalks

We will now apply our previous definition to a special kind of schemes, where the Étale structure is very easy.

Definition 2.6. Let $\Phi : x = \text{Spec } \overline{k} \to X$ be a geometric point, and let $\mathcal{F}$ be a sheaf of Abelian groups on X_{et}. Then the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at x is given by $\Phi^{-1}(\mathcal{F})$.

This has has a very convenient property of determining exactness:

Theorem 2.7. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of abelian groups on X_{et}. The following are equivalent:*

(1) *The sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact

(2) *The sequence*

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$$

is exact for each geometric point $x \to X$.

Now lets take a look at the category of Abelian groups on x_{et}, with $x = \text{Spec } \overline{k}$. We know that all Étale coverings are in the form $\coprod_i x \to x$, and morphisms between the coverings are nothing but maps between the index sets. As such, we get the following result:

Lemma 2.8. *All sheaves of Abelian groups x_{et} are of the form $\coprod_{i \in I} \Omega \mapsto x^I$. As a result, we can identify the category $\boldsymbol{Ab}_{x_{et}}$ with $\boldsymbol{Ab}$, by looking at the section of $x \to x$. Likewise, if $\mathcal{C}$ is a sheaf of rings of x_{et}, then $\boldsymbol{Mod}_{\mathcal{C}} \cong \boldsymbol{Mod}_R$, with $R = \mathcal{C}(x \to x)$.*

So stalks can now been seen as Abelian groups as well. The following lemma gives an explicit description of them, which is also useful for proving Theorem 2.7.

Lemma 2.9. *Let $x \to X$ be a geometric point, and let $\mathcal{F}$ be a sheaf on X_{et}. Then, under above identification, $\mathcal{F}_x$ is the set of triplets $[s, U, \Phi]$, where $U \to X$ is an Étale morphism, $s \in \mathcal{F}(U)$, and $\Phi : x \to U$ is an arrow compatible with $x \to X$ and $U \to X$. We have that $[s, U, \Phi] = [t, V, \psi]$ is there exists a $W \to U$ and $W \to V$, along with a compatible $x \to W$ such that $s|_W = t|_W$.*

2.3 Existence of injectives

As with the case of topological spaces, we will rely on the following result of the Commutative algebra.

Lemma 2.10. *Let R be a ring. Then $\boldsymbol{Mod}_R$ has enough injectives.*

This finally allows us to prove the existence of injectives, immitating Hartshorn's proof for the topological case.

Theorem 2.11. *Let X be a scheme, and let $\mathcal{C}$ be a sheaf of rings. Then the category $\boldsymbol{Mod}_{X_{et}}$ has enough injectives.*

Proof. Let $\mathcal{F}$ be a $\mathcal{C}$-module. Let $f_x : x \to X$ be a geometric point. From Lemma 2.9 and Lemma 2.10, it follows that $\mathbf{Mod}_{\mathcal{C}_x}$ has enough injectives. So let $\mathcal{F}_x \to I_x$ be an embedding into an injective object. This induces a morphism $\mathcal{F} \to f_{x*}(I_x)$ by Lemma 2.5. Let $\mathcal{I} = \prod_x f_{x*}(I_x)$. We will prove that $\mathcal{I}$ is injective, and that the induced morphism $\mathcal{F} \to \mathcal{I}$ is also injective. We will start proving that $\mathcal{I}$ is injective. Since products of injective objects are alwas injective, it suffices to prove that $f_{x*}(I_x)$ is injective for any geometric point $x \to X$. Let $0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{C}$-modules. We have to prove that the following sequence is exact.

$$0 \to \text{Hom } (\mathcal{K}, f_{x*}(I_x)) \to \text{Hom } (\mathcal{G}, f_{x*}(I_x)) \to \text{Hom } (\mathcal{H}, f_{x*}(I_x)) \to 0$$

But by Lemma 2.5, this is the same as saying that the following sequence is exact, for all x:

$$0 \to \text{Hom } (\mathcal{K}_x, I_x) \to \text{Hom } (\mathcal{G}_x, I_x) \to \text{Hom } (\mathcal{H}_x, I_x) \to 0$$

By Theorem 2.7, we know that $0 \to \mathcal{K}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is exact. As I_x is injective, above sequence is therefore exact as well. So $f_{x*}(I_x)$ is injective.
It remains to prove that $\mathcal{F} \to \mathcal{I}$ is injective. Now let $U \to X$ be Étale, and let $s \in \mathcal{F}(U)$ be nonzero. Let $\langle s \rangle \subseteq \mathcal{F}$ be the subsheaf generated by s. This is nonzero, so by Theorem 2.7, it follows that there exists a geometric point $x \to X$ such that $\langle s \rangle_x \neq 0$. The morphism $\langle s \rangle_x \to I_x$ is injective, so nonzero, so the morphism $\langle s \rangle \to f_{x*}(I_x)$ is nonzero as well. So we must have that $s \not\to 0$ by $\mathcal{F}(U) \to f_{x*}(I_x)(U)$, and therefore also by $\mathcal{F}(U) \to \mathcal{I}(U)$. This proves that $\mathcal{F} \to \mathcal{I}$ is injective. □

3 Exercises

Exercise 3.1. Prove Theorem 1.3.

Exercise 3.2. Prove that the sheafification of preserves equalizers and finite products, using our construction.

Exercise 3.3. Give an example of a (big) category, and a sheaf of presheafs (of sets) that is impossible to sheafify.

Exercise 3.4. Give an example of a ring R, a and a prime $\mathfrak{p} \in \text{Spec } R$ such that $\mathcal{O}_{\text{Spec } R, \mathfrak{p}} \neq R_{\mathfrak{p}}$. Here is $x = \text{Spec } \overline{\kappa_{\mathfrak{p}}} \to X$ the geometric point corresponding to $\mathfrak{p}$, and $\mathcal{O}_{\text{Spec } R}$ the sheaf of global sections.

Exercise 3.5. Is it possible to sheafify presheafs on small Étale sites using stalks (using Lemma 2.9 as a definition)?

Exercise 3.6. Prove Theorem 2.7. (*Hint:* Look at how this has been proven in the case of topological spaces. Use Lemma 2.9)

References

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