Constructible sheaves and the Base Change Theorem

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1 Constructible sheaves

1.1 Constructible sheaves

We will start with a definition and properties of contructible sheaves. For simplicity, we will assume X is a Noetherian scheme.

Definition 1.1. Let $\mathcal{F}$ be a sheaf of abelian groups on X_{et}. We call $\mathcal{F}$ *constructible* if there exists a finite surjective family of locally closed subschemes $f_i : X_i \to X$ such that $\mathcal{F}|_{X_i} = f_i^{-1}(\mathcal{F})$ is locally finite constant for all i.

Remark 1.2. If X where not Noetherian, then the family $X_i \to X$ had to be a family of *constructible* subschemes. But with Noetherian spaces, this is always the case.

Lemma 1.3. *The category of constructible sheaves is a strong Serre subcategory of $\boldsymbol{Ab}_X$. That is, for each exact sequence...*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

... the sheaf $\mathcal{G}$ is constructible if and only if $\mathcal{F}$ and $\mathcal{H}$ are.

Proof. Exercise. □

In the next couple of lemmas, we will develop an important class of constructible sheaves, what in fact are the building blocks of constructible sheaves.

Lemma 1.4. *Let $U \to X$ be an Étale morphism. Assume that U be quasi-separated and quasi-compact (Noetherian or affine, for example). Then there exists a finite family of locally closed subsheaves $X_i \to X$ such that $U \times_X X_i \to X_i$ is finite Étale.*

Proof. See Stacks Project, Chapter 50.69, Tag 095J. □

Lemma 1.5. *Let $f_* : U \to X$ be finite Étale. If $\mathcal{F}$ is a locally constant sheaf of finite rank on U_{et}, then so is $f_*(\mathcal{F})$.*

Proof. This follows from the fact that a finite Etale morphism is locally (w.r.t. the Etale topology) of the form $n.U \to U$ (see Stacks Project, tag 04HK) □

This gives us immediately a class of constructible sheaves.

Corollary 1.6. *For any $U \to X$ Étale, with U being quasi-separated and quasi-compact, the sheaf $j_{U,!}(\mathbb{Z}/n\mathbb{Z})$ is constructible.*

1.2 Torsion sheaves

This chapter shows that whenever we want to show a result about cohomology for constructible sheaves, that it suffices to show it for a rather limited class of constructible sheaves.

Definition 1.7. A sheaf $\mathcal{F}$ on X_{et} *torsion* if $\mathcal{F} = \operatorname{colim}_n \ker(n : \mathcal{F} \to \mathcal{F})$. That is, for each $U \to X$, we have that each element of $\mathcal{F}(U)$ is locally torsion.

Theorem 1.8. *Every torsion sheaf $\mathcal{F}$ is a filtered colimit of constructible sheaves.*

Proof. For each affine $U \to X$, let $z \in \mathcal{F}(U)$ be of finite order, say n. This yields a natural morphism $j_{U,!}(\mathbb{Z}/n\mathbb{Z}) \to \mathcal{F}$ mapping 1 to z on U. The image is constructible by Lemma 1.3, as it is a quotient of the source. If we add finite "unions" of those images (which are constructible by Lemma 1.3), we get a directed diagram of constructible sheaves. The limit is $\mathcal{F}$. □

Since cohomology commutes with filtered limits, it follows that we can compute cohomology for torsion sheaves out of cohomology of constructible sheaves. But we can make our job even easier! In order to do so, we have to use some results for abelian categories.

Definition 1.9. Let $\mathcal{C}$ be an Abelian category, and $T : \mathcal{C} \to \mathbf{Ab}$ be an additive functor. We call T *effaceable* if for each object $A \in \mathcal{C}$, and each $x \in T(A)$, there exists a monomorphism $u : A \to M$ such that $T(u)(x) = 0$.

Lemma 1.10. *The functor $H^q(X, -)$ on the category of constructible sheaves is effaceable for all $q > 0$.*

Proof. Let $\mathcal{F}$ be a constructible sheaf. Consider the Godement resolution $\mathcal{G} = \prod_\Omega i_*(\mathcal{F}_\Omega)$. This has natural embedding $\mathcal{F} \to \mathcal{G}$. As with skyscraper sheaves, we have that $H^q(i_*(\mathcal{F}_\Omega)) = 0$ and hence $H^q(\mathcal{G}) = 0$. Unfortunately, we don't know if $\mathcal{G}$ is constructible.
However, $\mathcal{G}$ torsion, so it is a filtered colimit of its constructible subsheaves.

Since $\mathcal{F}$ is also a contructible subsheaf, it follows that $\mathcal{G}$ is a filtered limit of constructible subsheaves containing $\mathcal{F}$. So looking at the construction of the colimit, $x \in H^q(\mathcal{F})$ will be mapped to 0 in some $H^q(\mathcal{H})$ for some constructible subsheaf $\mathcal{H} \subseteq \mathcal{G}$. So H^q is effaceable. □

Lemma 1.11. *Let $\mathcal{C}$ be an Abelian category. Let $T, T' : \mathcal{C} \to \boldsymbol{Ab}$ be two δ-functors, and let $\Psi : T \to T'$ be a morphism of δ-functors. Let $\xi \subseteq \mathcal{C}$ be a collection of objects, and suppose that each object of $\mathcal{C}$ is a subobject of an element of ξ. Also, suppose that T^q is effaceable for all $q > 0$. Then the following are equivalent:*

1. *The function $\Psi^q(A)$ is an isomorphism for all $q \geqslant 0, A \in \mathcal{C}$*
2. *The function $\Psi^0(M)$ is bijective, and $\Psi^q(M)$ is surjective for all $q \geqslant 0$, for all $M \in \xi$.*
3. *The function $\Psi^0(A)$ is bijective for all $A \in \mathcal{C}$, and T'^q is also effaceable for all $q \geqslant 0$.*

Before we head into the nice result, we need an alternate definition.

Lemma 1.12. *(Alternate definition for constructible sheaves) A sheaf $\mathcal{F}$ is constructible if there exists an embedding $\mathcal{F} \to \prod_i \pi_{i_*}(\mathbb{Z}/n_i\mathbb{Z})$, with $X_i \to X$ a finite surjective family of finite morphisms.*

Proof. Omitted. □

Corollary 1.13. *Let $X_0 \to X$ be an inclusion of schemes, and suppose that for all finite schemes $Y \to X$ the canonical map $H^q(X_0, \mathbb{Z}/n\mathbb{Z}) \to H^q(Y_0, \mathbb{Z}/n\mathbb{Z})$ is bijective for $q = 0$ and surjective for $q = 0$. Then for all constructible $\mathcal{F}$ on X, the canonical map $H^q(X, \mathcal{F}) \to H^q(X_0, i^{-1}(\mathcal{F}))$ is bijective.*

Proof. Take the category of constructible sheaves, and consider $T^q = H^q(X, -), T'^q = H^q(X_0, i^{-1}(\mathcal{F}))$, and let ξ be the collection of sheaves in the form $\pi_*(\mathbb{Z}/n\mathbb{Z})$, with $\pi : Y \to X$ being a finite morphism. Since f_* is exact if f is finite, we have $H^q(Y, \mathbb{Z}/n\mathbb{Z}) = H^q(X, \pi_*(\mathbb{Z}/n\mathbb{Z}))$ for all q and the same for X_0, Y_0. That sheaves in the form $\pi_*(\mathbb{Z}/n\mathbb{Z})$ are constructible follows from the alternate definition, and that any constructible sheaf can be embedded in products of those sheaves is equivalent to that definition. □

Naturally, with Theorem 1.8, this generalizes to arbitrary torsion sheaves!

2 Proper base change theorem

We want to prove the following:

Theorem 2.1. *Let $f : X \to S$ be a proper morphism of schemes, and let $g : S' \to S$ be any morphism of schemes. This gives us the following Cartesian diagram of arrows:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

Then for any torsion $\mathcal{F}$ on X_{et}, we have that the canonical map $R^q(g^{-1}f_*)(\mathcal{F}) \to R^q(f'_*g'^{-1})(\mathcal{F})$ is an isomorphism.

The proof is very long. As such, I will only show where constructible sheaves take place, so we can reduce to a simpler case.

2.1 Reduction to Special Fibres

Lemma 2.2. *It suffices to prove Theorem* 2.1 *for $\mathcal{F}$ constructible.*

Lemma 2.3. *It suffices to prove Theorem* 2.1 *under the assumption that S' is a geometric point (we call it Ω), and S is affine (from now on, we call $X_0 = X'$).*

Proof. Exercise. *Hint:* This relies on the fact that exactness can be checked on the stalks. □

For the following definition, it is convenient to restrict to a spaces case of affine rings, namely *strict Henselizations* of rings. Under an algebraic geometric standpoint, these are defined naturally:

Definition 2.4. Let R be a ring, and let $\Omega \to S := \text{Spec } R$ be a geometric point. The *strict Henselization* of R , called R_Ω at Ω is the stalk of the structure sheaf of R at Ω. This comes with a natural map $R \to R_\Omega$. A ring is *strictly Henselian* if it is the strict Henselization of a ring.

Those Henzelian rings pack nice properties, which go beyond the scope of this presentation.

Lemma 2.5. *It suffices to prove Theorem* 2.1 *under the assumption that, in addition to previous lemma, that S is Henselian, with Ω mapping to the closed point.*

Proof. Omitted. This relies on properties of Henzelian rings. □

The beauty lies in the fact that we are now able to make the description of the right derived functors more familiar. Since sheaves on Ω are nothing but abelian groups, we find that $f'_* : \mathbf{Ab}_{X_0} \to \mathbf{Ab}_\Omega$ coincides with the global section functor. But since S is the spectrum of the strict henzelisation of a ring at Ω, it follows that S has no nontrivial Étale neighborhoods at Ω. So the functor $g^{-1} : \mathbf{Ab}_S \to \mathbf{Ab}_{S'}$ is also the global section functor. In particular, we have that $g^{-1}f_* : \mathbf{Ab}_X \to \mathbf{Ab}_{S'}$ is a global section functor. Since g'^{-1} is exact, we find that the theorem reduces to:

Lemma 2.6. *In order to prove Theorem* 2.1*, it suffices to prove that* $H^q(X, \mathcal{F}) \to H^q(X_0, i^{-1}(\mathcal{F}))$ *is bijective. Here is* X_0 *the geometric fibre of the closed point* $\Omega \to S$*, where* $X \to S$ *is proper, and* S *is strictly Henselian.*

This is where the constructibility part comes in! With Theorem 1.8 and Corollary 1.13, we gain the following result:

Lemma 2.7. *In order to prove Theorem* 2.1*, it suffices to prove that* $H^q(X, \mathbb{Z}/n\mathbb{Z}) \to H^q(X_0, \mathbb{Z}/n\mathbb{Z})$ *is bijective for* $q = 0$*, and surjective for* $q > 0$*. Here is* X_0 *the geometric fibre of the closed point* $\Omega \to S$*, where* $X \to S$ *is proper, and* S *is strictly Henselian.*

2.2 Further steps (Sketch)

In order to prove the theorem further, we rely on Chow's lemma:

Lemma 2.8. *Suppose* X *is proper over* S*. Then there exists a scheme* $\overline{X} \to X$ *over* S *that is locally projective. The morphism* $\overline{X} \to X$ *is birational and surjective.*

The idea is to break X into a projective open $U \to X$, and a closed set of lower dimension. Doing this repeatedly, we may assume that $X \to S$ is locally projective. This assumption can be reduced the the assumption that the map is of the form $\mathbb{P}^n_S \to S$. From there, the assumption can be reduced to X being a curve by using the rational maps, and blow-ups. This puts us into the familiar terrain of curves; albeit over a strictly Henselian ring, as opposed to a field. From there, we have that $H^q(X_0, f^{-1}(\mathcal{F})) = 0$ for $q \geqslant 3$, so this leaves us to three cases to be threated separately.

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