Topics in Algebraic Geometry

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1 Introduction and motivation

In this talk i will give an incomplete and at sometimes imprecise introduction to the Picard/ Jacobian variety. Given a space say an algebraic curve, a variety ... the Picard variety is a space such that each point of it corresponds an an isomophism class of "line bundles" and similarly the Jacobian which you restrict to only degree zero line bundles. They have lots of more structure than just space and it is of great importance to find that these spaces exist and what they are. The theory over $k = \mathbb{C}$ is an old subject mostly work done by Abel, Jacobi and Riemann. By the work of Grothendieck et al it is reformulated and generilized greatly with vast conceptual power but the disadvantage is that is very hard to actually find these schemes. The Jacobi construction is a map

$$\{\text{ compact Riemann surfaces}\} \longrightarrow \{\text{Abelian varieties}\}$$

For a compact Riemann surface X of genus g the vector space of differential 1-forms $\Omega^1(X)$ is a vector space of dimention g and has a basis $(\omega_1, \ldots, \omega_g)$ Integration along cycles $[c] \in H_1(X, \mathbb{Z})$ gives a map $\int_{[c]} : \Omega^1(X) \to \mathbb{C}$ and we have

$$J(X) = \frac{\Omega^1(X)^*}{H_1(X, \mathbb{Z})} \cong \frac{\mathbb{C}^g}{\Lambda} \cong \frac{\mathbb{R}^{2g}}{\mathbb{Z}^{2g}} \cong \mathbb{T}^{2g}$$

It was the work of Abel, Jacobi that this is equivalent to the group of divisors of degree 0

2 Picard and its relation to cohomology

Suppose we are given a smooth, projective curve X over $k = \overline{k}$. Recall that a divisor is a finite sum $\sum n_i p_i \quad p_i \in X(k), n_i \in \mathbb{Z}$ and they form an abelian group $Div(X)$. The degree defined a homomorphism $deg : Div(X) \to \mathbb{Z}$. Given a rational function $f \in k(X)$ define the principal divisor as $div(f) = \sum ord_p(f)p$. The Picard group or the divisor class group is defined $\mathrm{Pic}(X) = \frac{Div(X)}{Princ(X)}$. The kernel $\ker(deg) = \mathrm{Pic}^0(X)$

Theorem 2.1. *There is an isomorphism* $\mathrm{Pic}(X) = H^1(X_{Zar}, \mathcal{O}_X^*)$

Proof. Let $p \in X$ and denote by $\mathbb{Z}_p$ the skyscraper sheaf with values in integers and the sheaf $k(X)^*$ which is constant $k(X)^*(U) = k(X)^*$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow k(X)^* \xrightarrow{div} \bigoplus_p \mathbb{Z}_p \longrightarrow 0$$

So we get a long exact sequence in cohomology

$$H^0(X, k(X)^*) \longrightarrow H^0(X, \bigoplus_p \mathbb{Z}_p) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, k(X)^*)$$

Now, the sheaf $k(X)^*$ is flasque(flabby) so its cohomology for $i > 0$ vanishes [HAG III]. Therefore the exact sequence becomes

$$k(X)^* \longrightarrow Div(X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow 0$$

And the result follows.

□

Remark 1. In fact for the sheaf $\mathbb{G}_m$ we have canonical identifications

$$H^1(X_{fppf}, \mathbb{G}_m) = H^1(X_{\acute{e}t}, \mathbb{G}_m) = H^1(X_{Zar}, \mathbb{G}_m) = \mathrm{Pic}(C) = H^1(X, \mathcal{O}_X).$$

After reformulating the skyscraper sheaf for the étale topology the proof is almost identical to the one i have above (for curves). [Stacks 03P8]. Also using the Kummer sequence that Prof.Bas introduced last time it is easy to find to what the cohomology groups $H^i(X_{\acute{e}t}, \boldsymbol{\mu}_n)$ of a curve are isomorphic to.

3 Construction of the Jacobian

Here a curve X means a separated scheme of finite type over $\mathrm{Spec}(k)$ which is also assumed to be smooth, projective and geometrically integral i.e. $X_{\overline{k}} = X \times_k \mathrm{Spec}(\overline{k})$

Definition 3.1. *Let C/k a curve with fixed point $x_0 \in C(k)$. The Jacobian of C is an abelian variety $J = \mathrm{Jac}(C)$ with a morphism $j : C \to J$ taking x_0 to 0 such that for any morphism $f : X \to A$ to an abelian variety with $f(x_0) = 0$ there is a unique $\tilde{f} : J \to A$ making the diagram commute*

$$\begin{array}{ccc} C & \xrightarrow{j} & J \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \tilde{f}} \\ & & A \end{array}$$

Since by defintion (J, j) is given by a universal property it is unique up to unique isomorphism if it exists. Also this functor is left adjoint to the forgetfull functor $\iota : \mathbf{AbVar}_k \to \mathbf{Var}_{k,*}$

$$\mathbf{Var}_{k,*} \underset{\iota}{\overset{j}{\rightleftarrows}} \mathbf{AbVar}_k$$

This definition is good but we there is another way that involves the Picard group.

Definition 3.2. *Let X be a scheme. An $\mathcal{O}_X$-module sheaf $\mathcal{L}$ is called an invertible sheaf or a line bundle on X if it is locally free of rank 1, so in other words there must exist and open covering $(U_i)_{i \in I}$ together with isomorphisms $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i},\ i \in I$.*

Theorem 3.1. *The isomorphism classes of invertible sheaves on a scheme X form a commutative group under the operation $(\mathcal{L}, \mathcal{L}') \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$. It is denoted by $\mathrm{Pic}(X)$ The inverse is given by by $\mathcal{L}^{-1} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ and it is functorial(contravariantly) in X.*

The Picard group unifies many geometric and algebraic notions which are seemingly unrelated and so actually computing it, is of great importance.

Example 3.1. Let $(A, \mathfrak{m})$ a local ring. Then $\mathrm{Pic}(A) = 0$. By the Serre-Swan theorem line bundles correspond to finitely generated , projective A-modules such that the rank function $\mathrm{Spec}(A) \to \mathbb{N}$ is constant with value 1. For every UFD it holds that $\mathrm{Pic}(A) = 0$ [Stacks 0AFW] and so since a PID is UFD it follows.

Example 3.2. If K is an algebaic number field i.e. a finite field extention of $\mathbb{Q}$ and $\mathcal{O}_K$ is its ring of algebraic integers then $\mathrm{Pic}(\mathrm{O}_K)$ is canonically isomorphic to the ideal class group.

Example 3.3. Every PID is a Dedekind domain. From the above we have that $\mathrm{Pic}(A)$ vanishes In fact a Dedekind domain is a principal ideal domain iff the Picard group is trivial.

Example 3.4. Picard group of affine line with doubled origin $\overline{\mathbb{A}^1}$. Cover $\overline{\mathbb{A}^1}$ by two copies of affine lines $\mathcal{U}_1 = \mathbb{A}^1, \mathcal{U}_2 = \mathbb{A}^1$ and $\mathbb{A}^1 \cap \mathbb{A}^1 = \mathbb{A}^1 \setminus \{0\}$ and we have the Cech complex

$$\Gamma(\mathcal{U}_1, \mathcal{O}^*_{\mathbb{A}^1}) \times \Gamma(\mathcal{U}_1, \mathcal{O}^*_{\mathbb{A}^1}) \longrightarrow \Gamma(\mathcal{U}_1 \cap \mathcal{U}_1, \mathcal{O}^*_{\mathbb{A}^1 \setminus \{0\}})$$

Since the ring of functions of $\mathbb{A}^1$ is $k[x]$ the units are k^*. The ring of functions of $\mathbb{A}^1 \setminus \{0\}$ is the localized ring $k[x]_{(x)}$ localized at the maximal ideal $\mathfrak{m} = (x)$ which is equal to $k[x, x^{-1}]$. Every polynomial $f \in k[x, x^{-1}]$ which has negative powers of x can be written as $f = x^{-n} g(x)$ for n a natural number with $g \in k[x]$ so $f^{-1} = x^n \frac{1}{g}$ so this imposes that $g \in k^*$ so the units are the functions of the form $\{ax^n \colon a \in k^*, n \in \mathbb{Z}\}$ The image of d^0 are the functions $\frac{a_1}{a_2}$ where $a_1, a_2 \in k^*$ and so the cokernel is isomorphic to the ring of integers $\mathbb{Z}$

Example 3.5. Every(!) abelian group is the Picard group of some Dedekind domain.

Let C be a curve with the above properties i.e. separated, of finite type over $\mathrm{Spec}(k)$, smooth, projective and geometrically integral or we can assume that it is smooth, projective variety over k. The reason for this is that there is no difference between Cartier divisors and Weil and so life is easier. If $D = \sum D_x x$ is a divisor on C the degree is defined as above. Since C is smooth there is a well defined map $\mathrm{Pic}(C) \to \mathbb{Z}$. For T an arbitrary scheme or variety we define $\mathrm{Pic}^0(C \times T)$ to be the subset of $\mathrm{Pic}(C \times T)$ consisting of invertible sheaves $\mathcal{L}$ with $deg(\mathcal{L}_t) = 0$ for all $t \in T$. The line bundles on $C \times T$ should be thought of as the line bundles on C parametrized by the space T. So we have that the following sequence is exact

$$0 \longrightarrow \mathrm{Pic}^0(C \times T) \longrightarrow \mathrm{Pic}(C \times T) \longrightarrow \prod_{t \in T} \mathbb{Z}$$

where the last map is $\mathcal{L} \mapsto (deg(\mathcal{L}_t))_{t \in T}$. We now define the functor $\mathrm{Pic}^0_C : \mathbf{Sch}_k^{op} \to \mathbf{Ab}$ by sending the scheme (or variety) T to $\mathrm{Pic}^0(C \times T)/\mathrm{Pic}(T)$. It parametrizes the degree zero line bundles of the fiber product modulo the trivial line bundles that are induced by the pullback. I will explain in the next section why we did this choice.

Sketch proof of the construction assuming that $C(k) \neq \emptyset$ and that $k = \overline{k}$. We want to construct a variety such that $J(k)$ is the group of divisor classes of degree zero in C. Let $r > 0$ and $C^r = C \times C \times \ldots C$ and we act on it by the group S_r the symmetric group and define the quotient space $C^{(r)} = C^r / S_r$.

Non trivial fact The set $C^{(r)}$ is a variety and non-singular. Maybe Andrea can help convicing us.

Let $\mathrm{Pic}^r(C)$ the set(it is not a group) of divisor classes of degree r. For a fixed point $P_0 \in C$ (we assumed that $C(k) \neq \emptyset$) the map $\mathrm{Pic}^0(C) \to \mathrm{Pic}^r(C)$ where $[D] \mapsto [D] + r[P_0]$ is a bijection so it suffices to find a variety representing $\mathrm{Pic}^r(C)$

Given a divisor of degree r the Riemann-Roch states that

$$\ell(D) = r + 1 - g + \ell(K - D)$$

where K is the canonical divisor. Since $deg(K) = 2g - 2$ if $deg(D) > 2g - 2$ then $deg(K - D) < 0$ and $\ell(K - D) = 0$ thus

$$\ell(D) = r - g + 1 > 0, \quad \text{if } r = deg(D) > 2g - 2.$$

In particular for large enough r in our case $r > 2g-2$ (it is $\geq$ than the genus g see [Edixhoven pp 30 8.1.3]) every divisor class of degree r contains an effective divisor and so the map

$$C^{(r)} \to \mathrm{Pic}^r(C)$$

is surjective.

If we could find a section of ϕ meaning a morphism $s : \mathrm{Pic}^r(C) \to C^{(r)}$ such that $\phi \circ s = \mathrm{id}$ then $s \circ \phi$ is a morphism $C^{(r)} \to C^{(r)}$ and it is a regular map (This fact i will not prove.)

And so by taking the pullback

$$\begin{array}{ccc} J' & \longrightarrow & C^{(r)} \\ \big\downarrow & & \big\downarrow {\scriptstyle (\mathrm{id}, s\circ\phi)} \\ C^{(r)} & \xrightarrow{\Delta} & C^{(r)} \times C^{(r)} \end{array}$$

then the map from $J'(k) = \{(a,b) \in C^{(r)} \times C^{(r)} \colon a = b, \quad b = s(\phi(a))\}$ to $\mathrm{Pic}^r(C)$ sending $b \mapsto \phi(b)$ is an isomorphism. This shows that Pic^r is represented by J' which is a closed subvariety of $C^{(r)}$ because Δ is a closed immersion(As a variety it is separated .)

The problem is that you can't find a natural way of associating such a section s ! Because for an invertible sheaf $\mathcal{L}$ of degree $r > deg(K) = 2g-2$ as is our case by Riemann -Roch the dimension of the vector space of effective divisors that is linear equivalent to it is $r-g$, see remark [HAG pp 296 IV 1.3.2.]. The problem was solved by Weil by heurestically it is done by finding small open subsets such that sections exist locally and then by patching. This is highly non-trivial. See [Edixhoven pp 46, Milne]

4

The reason that we defined the Picard functor as the line bundles of the product modulo the trivial ones that are induced by the pullback has the chance to be representable as if we dont mod out it is not representable. Recall that for an arbitrary scheme (separated and of finite ype) $f : X \to S$ we defined the absolute Picard functor as $\mathrm{Pic}_X(T) = \mathrm{Pic}(X_T)$ where $X_T := X \times_S T$ To see that this is not representable here is the following

Example 4.1. Take as the base scheme $S = \mathbb{P}^1$ so for any scheme X take the product $\mathbb{P}^1_X = X \times \mathbb{P}^1$ which carries the non-trivial line bundle $\mathcal{O}(1)$. If we pull back this line bundle alone the standard zariksi affine covering $\mathbb{A}^1 \coprod \mathbb{A}^1 \to \mathbb{P}^1$ it becomes trivial so in particular the morphism $\mathrm{Pic}(\mathbb{P}^1_X) \to \mathrm{Pic}(\mathbb{A}^1_X \coprod \mathbb{A}^1_X)$ is not injective. Since a representable functor is necesessary a sheaf for the Zariski topology it follows that the absolute functor is not representable.

So we can define the relative picard functor (presheaf) as above $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ and equipping the category $\mathbf{Sch}/S$ with a Grothendieck topology $\tau = \text{Zar, ét, fppf}$ it is possible to impose conditions on the structure morphism assuming that $f : X \to S$ is proper,flat and of finite presentation such that the relative Picard functor is representative by a scheme.

For example im quoting the following results.

Theorem 4.1.

1. *If f is flat, projective and with geometrically integral fibers then $\mathrm{Pic}_{X/S}$ is representable by a scheme locally of finite presentation and separated over S Grothendieck FGA*
2. *If f is flat projectve and geometricaly reduced fibers such that irreducible components of the fibers of f are geometrically irreducible then it is representable by a scheme locally of finite presentation*
3. *If $S = \mathrm{Spec}(k)$ for a field and f is proper then the functor is representable by a scheme that is separated and locally of finite type over k Murre-Oort*

Remark 2. Unlike the case of curves it is not always true that $\mathrm{Pic}_C(\overline{k})/\mathrm{Pic}^0_C(\overline{k}) = \mathbb{Z}$. We set $\mathrm{Pic}_C(\overline{k})/\mathrm{Pic}^0_C(\overline{k}) = \mathrm{NS}(X)$ the so-called *Neron-Severi* group.

Remark 3. If we weaken the assumptions the the functor might still be representable by a certain kind of "space" but not by a scheme.

Remark 4. In general it is really hard to find models of the Jacobian varieties except in the case when C is an elliptic curve where $J(X) \cong X$. This is far from useless because one can compute the group law in the jacobian easier than in the curve using chords and tangents.

5 Exercises

1. Verify the exactness of the sequence in theorem 2.1
2. Show that indeed for a scheme X, $\mathrm{Pic}(X)$ is a group and that it is contravariant functor from schemes to groups
3. Let C is an elliptic curve of genus 1 and fix a point P_0 show that the map $C \to \mathrm{Pic}^0(C)$, $P \mapsto$ divisor class of $(P)-(P_0)$ is a bijection of sets. It is actually an isomorphsim of groups. Can you find the inverse morphism ?

References

[1] Bosch et al Neron models

[2] Edixhoven, Moonen Van der Geer Abelian varieties

[3] Edixhoven notes on jacobians

[4] Hartsthorne Algebraic Geometry.

[5] Milne Abelian varieties

[6] Stacks project