Topics in Algebraic Geometry - Talk 3
The Étale Site

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1 Introduction and motivating examples

Today, we are going to talk about Grothendieck topologies. A Grothendieck topology is something that (generalizes/axiomatizes) the notion of an open cover of a topological space.

1.1 Set Theory

All the categories we work with will be small. If you really want to know how to get a small (and nice) category of schemes, here is a reference [Sta16, Tag 000H].

1.2 Sheaves on a topological space

The first motivating example, and the example to keep in mind during the rest of the lecture, is that of sheaves on a topological space. Let X be a topological space, then there is a partially ordered set of open sets $\mathbf{Op}(X)$, which can be viewed as a category. Recall that a presheaf on X is precisely a presheaf on this category $\mathbf{Op}(X)$, i.e., a contravariant functor $\mathbf{Op}(X) \to \mathbf{Set}$.

Let F be a presheaf, then we call F a sheaf if for every open set U and every open cover $\{U_\alpha\}_{\alpha \in A}$ of U the following holds:

- For all $s \in F(U)$ such that $s\big|_{U_\alpha} = t\big|_{U_\alpha}$ for all α then $s = t$;
- If $s_\alpha \in F(U_\alpha)$ for all α and $s_\alpha\big|_{U_{\alpha,\beta}} = s_\beta\big|_{U_{\alpha,\beta}}$ then there is an $s \in F(U)$ such that $s\big|_{U_\alpha} = s_\alpha$ (this s is unique by the first property).

Equivalently, the following sequence of sets is an equalizer

$$F(U) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} F(U_{\beta,\gamma}),$$

where the parallel arrows are defined as follows: For every pair (β, γ) we have maps

$$\begin{array}{ccc} \prod_{\alpha \in A} U_\alpha & \xrightarrow{p_\beta} & U_\beta \\ \downarrow{\scriptstyle p_\gamma} & & \downarrow \\ U_\gamma & \longrightarrow & U_{\beta,\gamma} \end{array}$$

which induces two maps into $\prod_{\beta,\gamma} F(U_{\beta,\gamma})$.The proof of this will be an exercise. In order to generalize this definition, we remark that the (fiber-) product of two open sets in the category $\mathbf{Op}(X)$ is precisely the intersection.

Sometimes, things that we expect/would like to be sheaves are not sheaves. To be more precise, if we have two sheaves of modules $\mathcal{F}, \mathcal{G}$ on a scheme, then their tensor product is in general not a sheaf. Of course there is a solution to this problem, which a functor which takes presheaves to sheaves!

We denote by $\mathrm{Pre}(X)$ the category of presheaves on X and by $\mathrm{Shv}(X)$ the category of sheaves (morphisms of sheaves are defined to be morphisms of presheaves). This means that we have a fully faithful functor $i : \mathrm{Shv}(X) \to \mathrm{Pre}(X)$. The sheafification functor L is the left-adjoint of this inclusion, i.e., there is a natural bijection (with $A \in \mathrm{Pre}(X)$ and $B \in \mathrm{Shv}(X)$)

$$\hom_{\mathrm{Pre}(X)}(A, i(B)) \leftrightarrow \hom_{\mathrm{Shv}(X)}(L(A), B).$$

Furthermore, sheafification commutes with finite limits([Sta16, Tag 00WJ]).

Now a real life example: Consider $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. If we sheafify we get $\mathcal{O}_{\mathbb{P}^1}$ but on global sections we get $\{0\} \otimes k^2 = 0$ which are not the global sections of $\mathcal{O}_{\mathbb{P}^1}$.

2 Definition and Examples of Grothendieck Topologies

Definition 1. *Let* $\mathbf{C}$ *be a category, then a Grothendieck topology on* $\mathbf{C}$ *is the data:*

- *For every object* $X \in \mathbf{C}$*, a set* $\tau(X)$ *whose elements are sets of morphisms* $\{U_\alpha \to X\}$*, we call such a set a cover of* X*.*

These should satisfy the following properties:

GI *If* $f : Y \to X$ *is an isomorphism, then* $\{f : Y \to X\}$ *is a cover of* X*.*

GII *If* $\{U_\alpha \to X\}$ *is a cover of* X *and* $\{V_{\alpha\beta} \to U_\alpha\}$ *are covers of* U_α *for every* α *then the collection of composites* $\{V_{\alpha\beta} \to X\}$ *is a cover of* X*.*

GIII *If* $f : Y \to X$ *is a morphism in* $\mathbf{C}$ *and* $\{U_\alpha \to X\}$ *is a cover of* X*, then*

$$Y \times_X U_\alpha$$

exists and $\{Y \times_X U_\alpha \to Y\}$ *is a cover of* Y*.*

A Grothendieck site is a small category $\mathbf{C}$ together with a Grothendieck topology on $\mathbf{C}$.

It will be an exercise to check that for a topological space X, the category $\mathbf{Op}(X)$ of open sets, together with open covers form a Grothendieck topology. This immediately gives us a Grothendieck topology on the category of opens of a scheme (consider the underlying topological space with the Zariski topology) with we will call the Zariski (Grothendieck-)topology on X.

It is good to note, that an open cover $\{U_i\}$ of a topological space X is 'the same' as a family jointly surjective open embeddings $\{U_i \to X\}$ (this will be made precise later). This then allows us to define a Grothendieck topology on the category of topological spaces. A cover of X is just a jointly surjective collection of open immersions. It is clear that the composition of an open immersion is again an open immersion, but some work has to be done for the fiber product. Note that **Top** is not a site since it is not a small category.

2.1 Examples of sites

Definition 2. *For X a scheme we define an (Zariski/étale) cover of X to be a family $\{U_\alpha \to X\}$ of jointly surjective open immersions/étale morphisms. If $\mathbf{C} = \mathbf{Sch}$ this defines a Grothendieck topology on $\mathbf{C}$.*

Remark 1. *As we saw last week, any flat morphisms of finite presentation is open, hence étale morphisms are open, which means any Zariski cover is an étale cover.*

Checking the axioms of a Grothendieck topology entails checking that these classes of morphisms contain isomorphisms and are closed under composition and fiber products.

- It is clear that the open immersions are closed under composition and contain isomorphisms. For base change see ([Vak] exercise 7.1.B).
- It is clear that isomorphisms are étale. Composition was one of the exercises last week, base change will be an exercise this week.

Of course, we might only be interested in the covers of a single scheme. So let $S \in \mathbf{Sch}$ be a scheme, then we want to define a category of 'coverings of S' which will replace the category of open sets on S.

As an intermediate step, we consider the comma category $\mathbf{Sch}/S$ with the comma-topology (see exercise 4). In this category, objects are maps $X \to S$ and morphisms are commuting triangles, covers of $X \to S$ are just covers of X. This is commonly called the big Zariski/étale site of X (depending on what topology we start with on **Sch**).

If we started with the Zariski topology, then we can also consider S_{Zariski}: Objects are open immersions $X \to S$ and morphisms are just morphisms of schemes over S (making the obvious triangle commute). Coverings are jointly surjective open immersions. This category is in fact equivalent to the category $\mathbf{Op}(S)$, the category of opens of the underlying topological space! (and the equivalence of categories 'preserves coverings')

Sketch of proof. Every open subset U of a scheme S determines a unique open subscheme $U \hookrightarrow S$ and if $V \subset U$ then there is a unique morphism $V \hookrightarrow U$. Reversely, every open immersion $X \to S$ induces an isomorphisms between X and an open subscheme of S. A morphism $U \to V$ induces an inclusion $U \subset V$ (thought of as open subschemes of S). $\square$

We can do the same construction with the étale topology on $\mathbf{Sch}/S$. We take the subcategory $S_{\text{étale}}$ of $\mathbf{Sch}/S$ of étale morphisms $X \to S$ which is usually called the small étale site over S (this name has nothing to do with small categories). In Table 1 we have collected the above examples.

Category	Objects	Morphisms	Coverings
Top	Topological spaces	Continuous maps	Topological covers
$\mathbf{Top}/X$	Continuous Maps $S \to X$	Continuous maps $/X$	Topological covers $/X$
X_{Top}	Open Immersions $S \to X$	Continuous maps $/X$	Topological covers $/X$
Sch	Schemes	Morphisms of schemes	Zariski covers
$\mathbf{Sch}/S$	Morphisms $Y \to S$	Morphisms of schemes $/S$	Zariski covers $/S$
S_{Zariski}	Open immersions $Y \to S$	Morphisms of schemes $/S$	Zariski covers $/S$
Sch	Schemes	Morphisms of schemes	Étale covers
$\mathbf{Sch}/S$	Morphisms $Y \to S$	Morphisms of schemes $/S$	Étale covers $/S$
$S_{\text{étale}}$	Étale maps $Y \to S$	Morphisms of schemes $/S$	Étale covers $/S$

Table 1: *In this table, we have collected a number of examples of sites, most of them related to schemes. If we replace the word étale by fppf we get three extra examples. For more, see [Sta16, Tag 020K].*

2.2 The small étale site of a field

Let k be a field, it might be interesting to consider the site $(\operatorname{Spec} k)_{\text{étale}}$. It turns out that we can completely classify schemes with an étale morphisms to $\operatorname{Spec} k$:

Proposition 1. *([Sta16, Tag 03PC]) Let k be a field. A morphism of schemes $U \to \operatorname{Spec}(k)$ is étale if and only if $\cong \coprod_{i \in I} \operatorname{Spec} k_i$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*

Proof. Omitted, makes for a nice exercise. □

2.3 Sheaves

Definition 3. *Let $\mathbf{C}$ be a category with a Grothendieck topology, a sheaf on $\mathbf{C}$ is a presheaf F such that for every cover $\{U_\alpha \to X\}$ the following is an equalizer diagram of sets:*

$$F(X) \longrightarrow \prod_{\alpha} F(\alpha) \rightrightarrows \prod_{\beta,\gamma} F(U_\beta \times_X U_\gamma).$$

- If we are working with a topological space, and $\mathbf{C}$ is the category of opens with the induced Grothendieck Topology then this is just the old notion of sheaf we had.
- If $\mathbf{C} = \mathbf{Sch}$ with the Zariski topology, then any scheme is a sheaf. What does this mean? If we have a scheme S, then the functor $\hom(-, S)$ is a sheaf. Which means that if we have a scheme X

with an open cover $\{U_\alpha\}_{\alpha \in A}$ and morphisms $\{U_\alpha \to S\}_{\alpha in A}$ that agree on overlaps, then there is a unique morphisms $X \to S$.

- More generally, if $\mathbf{C} = \mathbf{Sch}$ with the fpff topology, then any scheme is a sheaf! This is a theorem by Grothendieck, a readable proof is in [Vis05] (Theorem 2.55). Note that étale covers are fppf, hence schemes are sheaves for the étale topology.

More generally, we call a site *subcanonical* if every representable functor is a sheaf. In exercise 4/5 we will see that this notion is well behaved with respect to taking comma categories.

- On the site $S_{\text{étale}}$ we can define a functor to the category of rings sending an object

$$(f : X \to S) \mapsto \Gamma(X, \mathcal{O}_X),$$

Proving that this is a sheaf is not easy, and relies on the following result:

Proposition 2 ([Mil13] Proposition 6.6)**.** *In order to verify that a presheaf F on $S_{\acute{e}tale}$ is a sheaf, it suffices to check that F satisfies the sheaf condition for Zariski open coverings and for étale coverings $V \to U$ (consisting of a single map) with V and U both affine.*

- If $\mathcal{F}$ is a coherent sheaf of modules on S, then we can define a presheaf on $S_{\text{étale}}$ by

$$(f : X \to S) \mapsto \Gamma(X, f^*\mathcal{F})$$

and this can be shown to be a sheaf using the proposition above.

3 Sheafification

The following result is expected. The proof used smallness in an essential way. If there is enough time I might say something about it.

Proposition 3. *([Sta16, Tag 00W1]) Let C be a small category equipped with a Grothendieck topology, then the inclusion functor*

$$\text{Shv}(C) \overset{i}{\hookrightarrow} \text{Pre}(C)$$

has a left adjoint L (called sheafification), i.e., there is a natural bijection

$$\hom_{\text{Pre}(C)}(X, i(Y)) \leftrightarrow \hom_{\text{Shv}(C)}(L(X), Y).$$

Furthermore, sheafification commutes with finite limits.

References

[Mil13] James S. Milne. *Lectures on Etale Cohomology (v2.21)*, 2013. Available at www.jmilne.org/math/.

[Sta16] The Stacks Project Authors. *Stacks Project.* `http://stacks.math.columbia.edu`, 2016.

[Vak] Ravi Vakil. *MATH 216: Foundations of Algebraic Geometry.* `http://math.stanford.edu/~vakil/216blog/`.

[Vis05] Angelo Vistoli. *Grothendieck topologies, fibered categories and descent theory.* In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005. `http://homepage.sns.it/vistoli/descent.pdf`.

4 Exercises

1. *Let X be a topological space, show that a presheaf on X is a sheaf if and only if the following sequence of sets is an equalizer*

$$F(U) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} F(U_{\beta,\gamma})$$

(Hint: the first property means precisely that the map $F(U) \to \prod_\alpha F(U_\alpha)$ is injective, the second property means that $F(U)$ is isomorphic to the subset of $\prod_\alpha F(U_\alpha)$ where the parallel arrows agree.)

2. *Let X be a topological space, show that category $\mathbf{Op}(X)$ is a Grothendieck site where we take covers to be open covers.*

3. *Show that étale morphisms are preserved under base change. (Hint: The property of a morphisms being étale is local on the target so you can assume the target is affine.)*

4. *Let $\mathbf{C}$ be a site and let X be an object in $\mathbf{C}$. Consider the comma category $(\mathbf{C}/X)$ where the objects are arrows $C \to X$ and the morphisms commutative triangles*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

Let $A \to X$ be an object of $\mathbf{C}/X$, a collection of morphisms

$$\begin{array}{ccc} U_\alpha & \longrightarrow & A \\ & \searrow \quad \swarrow & \\ & X, & \end{array}$$

in $\mathbf{C}/X$ is called a covering if $\{U_\alpha \to A\}$ is a covering in the Grothendieck topology on $\mathbf{C}$. Show that this induces a Grothendieck topology on $(\mathbf{C}/X)$, it is called the slice topology.

5. *Prove the following Proposition (c.f. [Vis05], Proposition 2.59)*

Proposition 4. *If $\mathbf{C}$ is a sub-canonical site and $S \in \mathbf{C}$, then $(\mathbf{C}/S)$ is a sub-canonical site (with the slice topology).*

6. *Prove (or look at a proof of) Proposition 1*