The Étale Fundamental Group

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1 Profinite Groups

Definition 1. Let $X \in \mathbf{Top}$ be a topological space. We call X a *profinite* space if X is homeomorphic to a limit of a diagram in **Top** consisting of finite discrete spaces.

Lemma 2. *Let $X \in \mathbf{Top}$ be a topological space. The following are equivalent.*

1. *X is a profinite space.*
2. *X is Hausdorff, compact and totally disconnected.*

Proof. See [2, Tag 08ZY]. □

If $G \in \mathbf{Grp}$ is a group, consider the collection of all normal subgroups of finite index $\mathscr{H}$ of G. If $H, H' \in \mathscr{H}$, we define $H \leq H' \iff H' \subseteq H$ (note the reversal). This gives us a projective system $(G/H)_{H\in\mathscr{H}}$, and the limit in **Top**, where each G/H is endowed with the discrete topology, is denoted by

$$\widehat{G} := \lim_{H\in\mathscr{H}} G/H.$$

Definition 3. If G is a group, we call $\widehat{G}$ the *profinite completion.*

The object $\widehat{G}$ is a topological group. The multiplication in $\widehat{G}$ is given by $(g_H \cdot H)\cdot(g'_H \cdot H) = (g_H \cdot g'_H \cdot H)$. Moreover, the multiplication and inverse maps are continuous.

Example 4. The profinite completion of $\mathbb{Z}$ is $\widehat{\mathbb{Z}} = \lim_{n>0} \mathbb{Z}/n\mathbb{Z}$. Here, the projective system is partially ordered by divisibility of natural numbers. Explicitly, elements of $\widehat{\mathbb{Z}}$ are of the form $(a_n)_{n>0}$ such that $a_n \in \mathbb{Z}/n\mathbb{Z}$ for each $n > 0$ and where $a_n \equiv a_m \bmod m$ whenever $m \mid n$. If p is a prime number, then we recover the p-adic integers from the projective limit $\lim_{n>0} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$.

We have a natural homomorphism $G \to \widehat{G}$ given by $g \mapsto (g \cdot H)$.

2 Finite Étale Morphisms

Lemma 5. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent.*

1. *f is finite.*
2. *f is proper with finite fibres.*
3. *f is universally closed, separated, locally of finite type and has finite fibres.*

Proof. See [2, Tag 02LS]. □

If X is a scheme, we denote by $\mathcal{O}_X$ its structure sheaf. If $P \in X$ is a point, then P corresponds to some prime $P \in \operatorname{Spec} A$ for some open affine $\operatorname{Spec} A$. The stalk is denoted by $\mathcal{O}_{X,P}$. This is a local ring, and its residue field is denoted by $\kappa(P) := \mathcal{O}_{X,P}/P\mathcal{O}_{X,P}$.

Recall that if $f : X \to Y$ is a morphism, then the fibre of f at $P \in Y$ is given by pulling back; i.e. $X_P := X \times_Y \operatorname{Spec}(\kappa(P))$.

Theorem 6. *Let R be a ring and M an R-module. The following are equivalent.*

1. *M is a finitely generated projective R-module.*
2. *M is finitely presented and for every maximal ideal $\mathfrak{m} \subsetneq A$, the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is free.*
3. *There exists a collection $(f_i)_{i \in I}$ of elements of R such that $\sum_{i \in I} f_i \cdot R = R$ and each R_{f_i}-module M_{f_i} is free of finite rank.*

Proof. See [1, Theorem 4.6] □

Remark 7. The third condition in theorem 6 says that the sheaf associated to M on $\operatorname{Spec} R$ is locally free of finite rank.

Let M be a finitely generated projective R-module. From theorem 6 it follows that if we take a prime $\mathfrak{p} \in \operatorname{Spec} R$, then the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free. Moreover we can define a function

$$\operatorname{rank}_R(M) : \operatorname{Spec} R \to \mathbb{Z}$$

which maps a prime $\mathfrak{p}$ to the rank of $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. It is locally constant, hence continuous.

Remark 8. If $\operatorname{Spec} R$ is connected, for instance if R is a domain, then the rank function is constant.

Now let $f : X \to S$ be a finite and locally free morphism of schemes. Denote by $\mathrm{sp}(S)$ the underlying topological space of a scheme S. If $\operatorname{Spec} A$ is an open affine in S with $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$, then we obtain a continuous rank function

$$[B : A] : \mathrm{sp}(\operatorname{Spec} A) \to \mathbb{Z}$$

just as in the above construction. If these rank functions overlap on opens, then they coincide. So they glue. Hence they give rise to a continuous function on the underlying space of S. So we have a well-defined continuous map

$$\deg(f) : \mathrm{sp}(S) \to \mathbb{Z}.$$

Lemma 9. *Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.*

Proof. See [2, Tag 02GW] □

3 Construction of $\pi_1(S, \overline{s})$

Given any functor $F : \mathcal{C} \to \mathcal{D}$, the natural isomorphisms $F \to F$ give us a set $\operatorname{Aut}(F)$. Composition of natural transformations then endows $\operatorname{Aut}(F)$ with a group structure. If $\mathcal{D} = \mathbf{Set}$, then we obtain for every $X \in \mathcal{C}$ a natural left action

$$\operatorname{Aut}(F) \times FX \to FX, \qquad (\eta, x) \mapsto \eta_X(x). \tag{1}$$

Every $\operatorname{Aut}(FX)$ is also a group, hence $\prod_{X\in\mathcal{C}} \operatorname{Aut}(FX)$ is a group. Moreover, we have a natural group homomorphism

$$\operatorname{Aut}(F) \to \prod_{X\in\mathcal{C}} \operatorname{Aut}(FX), \qquad \eta \mapsto (\eta_X)_{X\in\mathcal{C}}\,. \tag{2}$$

If some $\eta \in \operatorname{Aut}(F)$ is in the kernel, then it is "pointwise" the identity and so η itself is the identity natural transformation. Hence the above map is an injection. Let us see what happens when we endow every FX with the discrete topology and every $\operatorname{Aut}(FX)$ with the compact-open topology. Recall that this means that for a given $\sigma \in \operatorname{Aut}(FX)$ a fundamental system of neighborhoods for σ is given by the sets

$$U_S(\sigma) := \{\tau : FX \to FX \mid \tau|_S = \sigma|_S\}\,,$$

where $S \subseteq FX$ is a *finite* subset. In the case that FX is finite, this is simply the discrete topology. This will be our interesting case.

The reason for choosing this rather complicated topology is that the action maps of eq. (1) become continuous maps.

Then $\prod \operatorname{Aut}(FX)$ becomes a topological space. Recall that a *profinite* space is by definition a topological space which is compact, Hausdorff and totally disconnected (the only connected components are the singletons).

Lemma 10. *The homomorphism from eq.* (2) *identifies* $\mathrm{Aut}(F)$ *with a closed subgroup of* $\prod \mathrm{Aut}(FX)$*. In particular, if every* FX *is finite, then* $\mathrm{Aut}(F)$ *is a profinite group.*

Proof. See [2, Tag 0BMR] $\square$

Fix a scheme S and an algebraically closed field Ω. By $\mathbf{Fet}_S$ we denote the category whose objects are finite étale morphisms $X \to S$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array} .$$

Now fix a geometric point $\overline{s} : \mathrm{Spec}(\Omega) \to S$. Let $X_{\overline{s}} := X \times_S \mathrm{Spec}(\Omega)$. Denote by $\mathrm{Fib}_{\overline{s}}(X)$ the underlying set of the scheme $X_{\overline{s}}$. Given a morphism $X \to Y$ in $\mathbf{Fet}_S$, we obtain a set-theoretic function $\mathrm{Fib}_{\overline{s}}(X) \to \mathrm{Fib}_{\overline{s}}(Y)$. We have thus found a functor

$$\mathrm{Fib}_{\overline{s}} : \mathbf{Fet}_S \to \mathbf{Set}.$$

Let's call it the *fibre functor at the geometric point* $\overline{s}$. By the above, we may enrich this functor as

$$\mathrm{Fib}_{\overline{s}} : \mathbf{Fet}_S \to \mathrm{Aut}(\mathrm{Fib}_{\overline{s}})\text{-}\mathbf{set},$$

where **set** denotes the category of *finite* sets (not to be confused with **Set**), and $\mathrm{Aut}(\mathrm{Fib}_{\overline{s}})$-**set** denotes the category of finite sets equipped with a continuous $\mathrm{Aut}(\mathrm{Fib}_{\overline{s}})$ action.

Definition 11. Given a connected scheme S and a geometric point $\overline{s} : \mathrm{Spec}(\Omega) \to S$, the *étale fundamental group* $\pi_1(S, \overline{s})$ is defined to be the automorphism group $\mathrm{Aut}(\mathrm{Fib}_{\overline{s}})$ of the fibre functor $\mathrm{Fib}_{\overline{s}}$.

Lemma 12. *Let* $Y \xrightarrow{f} X \xrightarrow{g} S$ *be morphisms of schemes. If* $g \circ f$ *is finite and* g *is separated, then* f *is finite. If moreover* $g \circ f$ *and* g *are finite étale, then so is* f*.*

Proof. See [3, Lemma 5.3.2], or one of Bas' exercises from the previous week. $\square$

The following proposition plays a key role.

Proposition 13. *If* $Z \to S$ *is a connected* S*-scheme and* $f, g : Z \to X$ *are two* S*-morphisms to a finite étale* S*-scheme* X *with* $f \circ \overline{z} = g \circ \overline{z}$ *for some geometric point* $\overline{z} : \mathrm{Spec}(\Omega) \to Z$*, then* $f = g$*.*

Proof. See [3, Corollary 5.3.3]. □

Given a morphism of schemes $f : X \to S$, define $\mathrm{Aut}_S(X)$ to be the group of scheme automorphisms of X preserving f. Our convention is that automorphisms act from the left. For a geometric point $\overline{s} : \mathrm{Spec}(\Omega) \to S$ there is a natural left action of $\mathrm{Aut}_S(X)$ on the geometric fibre $X_{\overline{s}} = X \times_S \mathrm{Spec}(\Omega)$ coming from base change from its action on X.

Proposition 14. *If $f : X \to S$ is a connected finite étale morphism, then the nontrivial elements of $\mathrm{Aut}_S(X)$ act without fixed points on each geometric fiber. As a consequence, $\mathrm{Aut}_S(X)$ is finite.*

Proof. See [3, Corollary 5.3.4]. □

Definition 15. Let $f : X \to S$ be a connected finite étale morphism. Then we call f a *Galois* cover if $\mathrm{Aut}_S(X)$ acts transitively on geometric fibres.

The topological group $\pi_1(S, \overline{s})$ is profinite. For every $X \to S$ in $\mathbf{Fet}_S$, The action of $\pi_1(S, \overline{s})$ on $\mathrm{Fib}_{\overline{s}}(X)$ is continuous.

Theorem 16. *Let S be a connected scheme and $\overline{s} : \mathrm{Spec}(\Omega) \to S$ a geometric point. The functor*

$$\mathbf{Fet}_S \to \pi_1(S, \overline{s})\text{-}\mathbf{set}, \qquad (X \to S) \mapsto \mathrm{Fib}_{\overline{s}}(X)$$

is an equivalence of categories. Here, connected finite étale morphisms correspond to sets with transitive $\pi_1(S, \overline{s})$ action and Galois covers to finite quotients of $\pi_1(S, \overline{s})$.

Proof. See [3, Theorem 5.4.2]. The proof is three pages later. □

Unfortunately, the fibre functor $\mathrm{Fib}_{\overline{s}}$ is not representable. However, it is so-called *pro-representable.* This means that there exists an inverse system $\widetilde{S} := (S_i)_{i \in I}$ of finite étale morphisms $f_i : S_i \to S$ indexed by a directed partially ordered set I and for each finite étale morphism $g : X \to S$ an isomorphism

$$\mathrm{Fib}_{\overline{s}}(X) \cong \lim_{\to i \in I} \mathrm{Hom}(S_i, X),$$

functorially in X.

Proposition 17. *The fibre functor $\mathrm{Fib}_{\overline{s}}$ is pro-representable.*

Proof. [3, Proposition 5.4.6]. □

We call $\widetilde{S}$ "the" universal covering scheme of S. It is possible to choose $\widetilde{S}$ so that each S_i is Galois over S. If the action is transitive, then the degree of the morphism is equal to the order of $\mathrm{Aut}_S(S_i)$. So we have another way to determine $\pi_1(S, \overline{s})$, namely via the pro-representability

$$\pi_1(S, \overline{s}) = \mathrm{Aut}_S(\widetilde{S}) := \lim_{\leftarrow i \in I} \mathrm{Aut}_S(S_i).$$

4 Examples

Example 18. Let k be any field and let $S = \mathrm{Spec}(k)$. Then a connected finite étale morphism $X \to S$ corresponds to a separable field extension $k \to L$. This is [2, Tag 00U3]. The choice of a geometric point $\overline{s}$ amounts to the choice of an algebraically closed field Ω. For a geometric point $\overline{s}$ the fibre functor maps a connected cover $X = \mathrm{Spec}\, L$ to the underlying set of $X \times_S \mathrm{Spec}(\Omega) = \mathrm{Spec}(L \otimes_k \Omega)$. This is a finite set indexed by the k-algebra homomorphisms $L \to \Omega$. But the image of each such homomorphism lies in the separable closure $k^{sep} \subset \Omega$. Let $\widetilde{k}$ be a projective system consisting of finite separable field extensions $(L_i)_{i \in I}$. Then $\widetilde{k}$ pro-represents $\mathrm{Fib}_{\overline{s}}$, that is

$$\pi_1(S, \overline{s}) = \mathrm{Aut}_k(\widetilde{k}) = \lim_{\leftarrow i \in I} \mathrm{Gal}(k_i/k) = \mathrm{Gal}(k^{sep}/k).$$

Example 19 (IV.2.5.3 of Hartshorne)**.** Let k be an algebraically closed field of characteristic 0. We will show that $\pi_1(\mathrm{Spec}\, \mathbb{P}^1_k, 0) = 1$. Let $f : X \to \mathbb{P}^1_k$ be a finite etale morphism. We may assume that X is connected. By base change and smoothness of f, the scheme X is smooth over k. Moreover X is proper over k because f is finite. Hence X is a curve. Using now that f is separable, we may apply Hurwitz' theorem. The theorem states that

$$2g(X) - 2 = \deg(f) \cdot \big(2g(\mathbb{P}^1_k) - 2\big) + \deg R,$$

where

$$R = \sum_{P \in X} \mathrm{length}(\Omega_{X/\mathbb{P}^1_k})_P \cdot P$$

is the "ramification divisor". Again since f is étale, f is unramified. Therefore $\deg R = 0$. Moreover the genus of $\mathbb{P}^1_k$ is zero. So the formula reduces to

$$2g(X) - 2 = -2\deg(f).$$

The only possible solution is $g(X) = 0$, $\deg(f) = 1$. Hence $X \cong \mathbb{P}^1_k$ and f is an isomorphism. Every finite étale covering is trivial, so the fundamental group is trivial.

Example 20. Let k be an algebraically closed field and set $S = \mathbb{A}^1_k \setminus \{0\}$. In the topological case, ($k = \mathbb{C}$), we would have a universal cover $\mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto e^z$. But there is no such finite étale map! We do have the finite étales given by $f_n : S \to S$, $z \mapsto z^n$ for a given $n \in \mathbb{Z}_{>0}$, with $\deg(f_n) = n$. This defines a projective system. We have

$$\mathrm{Aut}_S\left(S \xrightarrow{f_n} S\right) = \mu_n(k),$$

the group of n-th roots of unity in k. If $\zeta \in \mu_n(k)$ is a primitive n-th root of unity, then the action is given by $z \mapsto \zeta z$. So we find

$$\pi_1(\mathbb{A}^1 \setminus \{0\}, \overline{s}) = \lim_{\leftarrow n} \mu_n(k) \cong \widehat{\mathbb{Z}}.$$

5 Exercises

Exercise 1. Determine which of the following morphisms are etale.

1. The normalization $\mathbb{C}[t] \to \mathbb{C}[x,y]/(y^2 - x^3)$. Recall that the normalization is given by $t \mapsto (t^2, t^3)$.
2. $\mathbb{C}[t] \to \mathbb{C}[x,y]/(y^2 - x^2 - x^3)$ given by $t \mapsto (t^2 - 1, t^3 - t)$.
3. $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ given by $(x, y) \mapsto (x, x \cdot y)$.
4. The inclusion $\mathbb{C}[t^2, t^3] \to \mathbb{C}[t]$.
5. The inclusion $\mathbb{Q} \to \mathbb{Q}[x]/(x^2 - 4)$.
6. The inclusion $\mathbb{Q} \to \mathbb{Q}[x]/(x^2 + 4)$.
7. The inclusion $\mathbb{Q} \to \mathbb{Q}[x]/(x^2)$.
8. The inclusion $\mathbb{F}_3 \to \mathbb{F}_9$.

Exercise 2. Show that $\mathbb{A}^1_{\mathbb{Q}(\sqrt[3]{2})} \to \mathbb{A}^1_{\mathbb{Q}}$ is not etale.

Exercise 3. Let $A = \mathbb{C}[x] \to \mathbb{C}[x,y]/(y - x^2) = B$ be the inclusion. Show that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is etale everywhere except at one point.

Exercise 4. Let $X = \operatorname{Spec}(\mathbb{C}[x,y]/(f))$, where $f = y^3 - 3y + 2x$ and let $Y = \mathbb{A}^1_{\mathbb{C}}$. Show that the projection $X \to Y$ is etale everywhere except at two points.

Exercise 5. Let k be a field and n a positive integer. Use the Jacobian criterion to show that $\mathbb{A}^1_k \to \mathbb{A}^1_k$, $x \mapsto x^n$ is étale at no point of $\mathbb{A}^1_k$ if $\operatorname{char}(k) \mid n$ and otherwise it is étale at all $x \neq 0$.

Exercise 6. Show that the affine line over an algebraically closed field has trivial fundamental group.

References

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[3] Tamász Szamuely. *Galois Groups and Fundamental Groups*. 2010.