Uitwerkingen werkcollege 10.

7.4.3 (1) If $\dim U = 1$, then U is generated by one nonzero element, say a. This means that $U = L(a)$, so U is indeed a line.

(2) Suppose $\dim U = n-1$ and let $B = (v_1, v_2, \ldots, v_{n-1})$ be a basis. Let A be the $(n-1)\times n$ matrix with the vectors $v_1, \ldots, v_{n-1}$ as rows, and let A' be a row echelon form of A. The number of nonzero rows of A' is the dimension of the row space $R(A') = R(A) = U$, so all rows of A' are nonzero and there are $n-1$ pivots. Therefore, there is only one column without a pivot, which means that the kernel $\ker A' = \ker A = U^\perp$ is generated by one nonzero element, say a. This gives $U^\perp = L(a)$, and hence $U \subset (U^\perp)^\perp = L(a)^\perp = a^\perp$ by Proposition 3.33, part (3). By Example 7.58, the hyperplane $a^\perp$ has dimension $\dim a^\perp = n-1 = \dim U$, so from Lemma 7.54 we conclude $U = a^\perp$.

7.4.7 We weten dan U_1+U_2 een deelruimte is van V en dat dus $\dim(U_1+U_2) \leq 10$. Stelling 7.56 geeft dan:

$$\begin{aligned}\dim(U_1 \cap U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1+U_2)\\ &= 6 + 7 - \dim(U_1+U_2) \geq 13 - 10 = 3.\end{aligned}$$

8.2.1 5.5.4 Rotation over α is an isomorphism: its inverse is rotation over $-\alpha$. So the image of ρ is $\mathbb{R}^2$, so the rank of ρ, and thus of the associated matrices, is 2.

5.5.5 The ranks are 2, 2, 3, 2, and 2, respectively (see Exercise 6.2.1).

8.2.4 Set $U = L(S)$. By Proposition 3.33, we have $S^\perp = L(S)^\perp = U^\perp$. Hence, this follows from Proposition 8.20.

8.2.6 (1) $\mathrm{rk}(AB) = \dim \mathrm{im}(AB)$, en $\mathrm{im}(AB)$ is een deelruimte van $\mathrm{im}(A)$. Lemma 7.55 geeft dat $\mathrm{rk}(AB) \leq \mathrm{rk}(A)$. Als $\mathrm{rk}(B) = m$, dan is $B\colon F^n \to F^m$ surjectief, dus geldt $\mathrm{im}(AB) = A(B(F^n)) = A(F^m) = \mathrm{im}(A)$, dus $\mathrm{rk}(AB) = \mathrm{rk}(A)$. Om de 'not only if' te laten zien, willen we een voorbeeld waarin $\mathrm{rk}(AB) = \mathrm{rk}(A)$ en $\mathrm{rk}(B) < m$. We nemen $F = \mathbb{Q}$, $n = m = l = 1$, en $A = 0$ en $B = 0$.

(2) We hebben dat $\ker(B)$ een deelruimte is van $\ker(AB)$, dus $\dim(\ker(B)) \leq \dim(\ker(AB))$. De rangstelling (Stelling 8.3) geeft dan

$$\mathrm{rk}(B) = n - \dim(\ker(B)) \geq n - \dim(\ker(AB)) = \mathrm{rk}(AB).$$

Als $\mathrm{rk}(A) = m$ dan is A injectief vanwege de rangstelling en dan geldt $\ker(AB) = \ker(B)$. Voor de 'not only if' nemen we hetzelfde voorbeeld als in (1).

(3) Dit volgt uit (1).

(4) Dit volgt uit (2).

8.3.1 Since $V^\perp$ is generated by x_3 and x_5, we find

$$U \cap V = U \cap (V^\perp)^\perp = \{u \in U \,:\, \langle u, x_3 \rangle = \langle u, x_5 \rangle = 0\}.$$

Every $u \in U$ can be written as $u = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. As in Example 8.28, the equations $\langle u, x_3 \rangle = \langle u, x_5 \rangle = 0$ are then equivalent to $(\lambda_1, \lambda_2, \lambda_3)$ lying in the kernel of the matrix

$$\begin{pmatrix} \langle u_1, x_3 \rangle & \langle u_2, x_3 \rangle & \langle u_2, x_3 \rangle \\ \langle u_1, x_5 \rangle & \langle u_2, x_5 \rangle & \langle u_2, x_5 \rangle \end{pmatrix} = \begin{pmatrix} 7 & 7 & -3 \\ 7 & 7 & -3 \end{pmatrix}.$$

Its kernel is generated by $(-1, 1, 0)$ and $(3, 0, 7)$, which correspond to the vectors $-u_1 + u_2$ and $3u_1 + 7u_3$, so these two vectors generate $U \cap V$.

8.3.2 The intersection is generated by $(1, 0, 0, 1) \in F^4$.