Uitwerkingen werkcollege 11.

8.4.1

$$\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ -5 & -4 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -3 & -2 & -2 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 & -1 & 2 \\ -9 & 1 & 3 & -8 \\ 8 & -1 & -3 & 7 \\ -8 & 1 & 3 & -8 \end{pmatrix}.$$

8.4.4 If AB is invertible, then $f_{AB} = f_A \circ f_B \colon F^n \to F^n$ is a bijection. Its injectivity implies that $f_B \colon F^n \to F^n$ is injective and its surjectivity implies that that f_A is surjective. From Corollary 8.5 we then conclude that f_A and f_B are isomorphisms as well, so A and B are invertible as well.

8.4.5 Since I_n is invertible, it follows from Exercise 8.4.4 that M and N are invertible. Note that from $MN = I_n$, we obtain $f_M \circ f_N = \mathrm{id}$, so f_N is a right inverse of the isomorphism f_M. Since every right inverse of a bijection is equal to the inverse of that bijection (see Appendix A), we find that f_N is the inverse of f_M and hence $N = M^{-1}$. It follows that we also have $NM = M^{-1}M = I_n$.

9.1.2 We berekenen de beelden van de elementen van de begin-basis B en schrijven die uit als lineaire combinaties van de elementen van de eind-basis B.

$$\begin{aligned}
T(1) &= 3 = 3{\cdot}1 + 0{\cdot}x + 0{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\
T(x) &= 3x = 0{\cdot}1 + 3{\cdot}x + 0{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\
T(x^2) &= 3x^2 + (x-2){\cdot}2 = -4{\cdot}1 + 2{\cdot}x + 3{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\
T(x^3) &= 3x^3 + (x-2){\cdot}6x = 0{\cdot}1 - 12{\cdot}x + 6{\cdot}x^2 + 3{\cdot}x^3 + 0{\cdot}x^4 \\
T(x^4) &= 3x^4 + (x-2){\cdot}12x^2 = 0{\cdot}1 + 0{\cdot}x - 24{\cdot}x^2 + 12{\cdot}x^3 + 3{\cdot}x^4
\end{aligned}$$

Dus

$$[T]_B^B = \begin{pmatrix} 3 & 0 & -4 & 0 & 0 \\ 0 & 3 & 2 & -12 & 0 \\ 0 & 0 & 3 & 6 & -24 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

9.1.4 We geven allee n de uitkomst:

$$\begin{pmatrix} 3 & 0 & 7 & 0 \\ 0 & 3 & 0 & 7 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 5 \\ 8 & 0 & 2 & 0 \\ 0 & 8 & 0 & 2 \end{pmatrix}$$

9.2.1 (1)

$$M = [\mathrm{id}]_B^{B'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \text{and} \qquad N = M^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Of course, instead of finding the inverse of M, one can also find N by noticing that $v_1 = v_1'$ and $iv_i = v_i' - v_{i-1}'$ for $2 \le i \le 4$.

(2) we have $f_M = \varphi_B^{-1} \circ \mathrm{id} \circ \varphi_{B'} = \varphi_B^{-1} \circ \varphi_{B'}$, so for every $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we find

$$Mx = f_M(x) = \varphi_B^{-1}(\varphi_{B'}(x)) = \varphi_B^{-1}(x_1 v_1' + \ldots + x_4 v_4') = (x_1 v_1' + \ldots + x_4 v_4')_{B'}.$$

(3) Same as (2) with the roles of M and N reversed (as well as the roles of v_i and v_i').

9.2.2

$$[\mathrm{id}]_E^C = \begin{pmatrix} -1 & -2 & 1 \\ -2 & 1 & -1 \\ 0 & 3 & -2 \end{pmatrix} \qquad \text{and} \qquad [\mathrm{id}]_C^E = ([\mathrm{id}]_E^C)^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -4 & 2 & -3 \\ -6 & 3 & -5 \end{pmatrix}.$$

9.3.1 (1)

$$[T]_{E_3}^{E_2} = \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$$

(2)

$$\begin{aligned} [T]_C^B &= [\mathrm{id}]_C^{E_3} \cdot [T]_{E_3}^{E_2} \cdot [T]_{E_2}^B \\ &= \begin{pmatrix} 1 & -1 & 1 \\ -4 & 2 & -3 \\ -6 & 3 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 4 \\ -39 & -9 \\ -60 & -15 \end{pmatrix} \end{aligned}$$

9.4.2 (1) The line equals $L = L(a)$ with $a = (1, 2)$. The projection of e_i on L equals $\frac{\langle a, e_i \rangle}{\langle a, a \rangle} \cdot a$, so we have $\pi(e_1) = \frac{1}{5}a$ and $\pi(e_2) = \frac{2}{5}a$, so

$$[\pi]_B^B = \tfrac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

(2) Take $v_1 = a$ and $v_2 = (2, -1)$. Then

$$[\pi]_C^C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(3)

$$\begin{aligned} [\pi]_B^B &= [\mathrm{id}]_B^C \cdot [\pi]_C^C \cdot [\mathrm{id}]_C^B = [\mathrm{id}]_B^C \cdot [\pi]_C^C \cdot ([\mathrm{id}]_B^C)^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \tfrac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{aligned}$$