Uitwerkingen werkcollege 13.

10.2.1 The determinants are -4, 12, -16, 17, -10, respectively.
10.4.1 (1) -19, (2) 1, (3) 0, (4) 0.
10.4.4 (1) Voor iedere i hebben we $M_n{\cdot}e_i = \sum_{j\neq i} e_j$, want de i-de kolom van M_n heeft overal een 1 behalve op de i-de coördinaat. Dus voor $i \neq j$ geldt dat $M_n(e_i - e_j) = \sum_{k\neq i} e_k - \sum_{l\neq j} e_j = e_j - e_i$. Hieruit volgt het gevraagde meteen.
(2) $M_n(e_1 + \cdots + e_n) = \sum_i \sum_{j\neq i} e_j = (n-1)\sum_k e_k$, want iedere e_k komt $n-1$ maal voor.
(3) $(v_2, \ldots, v_n)$ is de basis van $v_1^\perp$ die we vinden door de vergelijking $\langle x, v_1\rangle = 0$ op te lossen.
(4) Dit volgt uit (1) en (2).
(5) We berekenen $\det(f)$ met behulp van de basis B. De matrix van f t.o.v. B is diagonaal, en de determinant is het product van de coëfficiënten op de diagonaal.

10.5.1 For the first three systems we have

$$A = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 2 & 2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \text{with} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

respectively. We can reduce the work by extending the matrix A with all three choices for b and finding a reduced row echelon form. The extended matrix is

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 1 & 1 \\ 3 & 2 & 2 & 0 & -1 & 1 \\ 0 & -1 & 2 & 0 & -1 & 1 \end{array}\right).$$

and it has reduced row echelon form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Hence the kernel of A is generated by $a = (-2, 2, 1)$, that is $\ker A = L(a)$. Now for the first b, namely $b = 0$, the solution set is $\ker A = L(a)$. For the second b, if we had extended A by only b, the last column of reduced row echelon form of this extension $(A|b)$ does not have a pivot in the last column, so the system is consistent. We obtain a solution by setting $x = (x_1, x_2, 1)$ and solving for x_1 and x_2, which gives $x = (-3, 3, 1)$. Therefore, the complete solution space is

$$\{(-3, 3, 1) + z \;:\; z \in \ker A\} = \{(-3, 3, 1) + \lambda a \;:\; \lambda \in \mathbb{R}\}.$$

For the third b, namely $b = (1, 1, 1)$, the reduced row echelon form of the extended matrix $(A|b)$ has a pivot in the last column, so there is no solution. For the last case, we have

$$A = \begin{pmatrix} 3 & 1 & 2 & -2 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

The extended matrix $(A|b)$ has reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

Hence, there is a unique solution, namely $x = (-13, 4, 16, -2)$.

10.5.2 (zonder (2))

(1) The determinant of C_a is $a^2 - a - 6 = (a-3)(a+2)$, so for $a \notin \{-2, 3\}$, we have $\det C_a \neq 0$, so C_a has rank 3. For $a \in \{-2, 3\}$, the rank of C_a is easily computed to be 2.

(3) If C_a is invertible, then the equation $C_A x = v_b$ has a unique solution, so if there is more than one solution, then we have $a = -2$ or $a = 3$. for $a = -2$, then extended matrix $(C_{-2}|v_b)$ is row equivalent with

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & b-4 \end{array}\right).$$

This shows that the system is consistent if and only if $b = 4$, so for $a = -2$ we only get the pair $(a, b) = (-2, 4)$ (and since $\ker C_{-2}$ is infinite, we do indeed get infinitely many solutions). For $a = 3$, the extended matrix $(C_3|v_b)$ is row equivalent with

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & b+2 \\ 0 & -3 & 7 & 3b+4 \\ 0 & 0 & 0 & b+1 \end{array}\right).$$

so the corresponding system is consistent if and only if $b = -1$, in which case we do indeed have infinitely many solutions.

(4) We have to look at the pair $(a, b) = (-2, 4)$ and as we have seen the extended matrix $(C_{-1}|v_4)$ is row equivalent with

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution set is therefore

$$\{(3, -3, 1) + \lambda(2, -1, 1) \ : \ \lambda \in \mathbb{R}\}.$$

11.1.2 For every $\lambda \in \mathbb{R}$, the function g given by $g(x) = e^{\lambda x}$ for all $x \in \mathbb{R}$ satisfies $(Dg)(x) = g'(x) = \lambda g(x)$, so $Dg = \lambda g$, so g is an eigenvector for eigenvalue λ.

11.1.3 Voor $\lambda \in \mathbb{R}$ hebben we

$$\det(A - \lambda \cdot I_2) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1).$$

De eigenwaarden van A zijn dus $\lambda_1 = -3$ en $\lambda_2 = 1$. Een basis van $E_{-3}(A)$ is $(1, 2)$. Een basis van $E_1(A)$ is $(1, 1)$. Om rekenfouten te vermijden is het een goede zaak om te controleren dat dit inderdaad eigenvectoren met de geclaimde eigenwaarden zijn.

De tweede matrix is op het college behandeld. De eigenwaarden daarvan zijn -3, 1 en 2, en de eigenruimten zijn 1-dimensionaal, met bases respectievelijk $(0, 0, 1)$, $(1, -1, 0)$ en $(2, -1, 0)$.