Uitwerkingen werkcollege 2.

1.4.6 Gegeven is dat $S \subset F^n$ en $0 \in S$. We moeten een equivalentie bewijzen. We bewijzen de twee implicaties 1 voor 1.

Neem aan dat $z \in S^\perp$. Volgens de definitie van $S^\perp$ betekent dit dat voor elke $x \in S$ geldt dat $\langle z, x\rangle = 0$. We moeten bewijzen dat z normaal is op S. Volgens de definitie van normaal betekent dat: voor alle p en q in S geldt dat $\langle z, q-p\rangle = 0$. Laat dan p en q in S zijn. Dan geldt:

$$\langle z, q-p\rangle = \langle z, q\rangle - \langle z, p\rangle = 0 - 0 = 0,$$

waarbij we hebben gebruikt: Propositie 1.4, $q \in S^\perp$ en $p \in S^\perp$.

Neem nu aan dat z normaal is op S. We moeten laten zien dat voor alle $x \in S$ geldt: $\langle z, x\rangle = 0$. Laat $x \in S$. Gegeven is dat $0 \in S$. Dus:

$$0 = \langle z, x-0\rangle = \langle z, x\rangle - \langle z, 0\rangle = \langle z, x\rangle - 0 = \langle z, x\rangle,$$

waarbij we hebben gebruikt: de definitie van normaal, en Propositie 1.4.

1.5.1 Any plane V through p_1, p_2, p_3 is given by

$$V = \{v \in \mathbb{R}^3 \ : \ \langle a, v\rangle = b\}$$

for some $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and some $b \in \mathbb{R}$. The fact that we have $p_i \in V$ for $1 \le i \le 3$ implies

$$\begin{aligned} b &= \langle a, p_1\rangle = a_1 + 2a_3, \\ b &= \langle a, p_2\rangle = -a_1 + 2a_2 + 2a_3, \\ b &= \langle a, p_3\rangle = a_1 + a_2 + a_3. \end{aligned}$$

Subtracting the first two equations yields $a_1 = a_2$. Subtracting the last two equations then gives $a_3 = 2a_1 - a_2 = a_1$. Any of the three equations then gives $b = 3a_1$, so the equation $\langle a, v\rangle = b$ is equivalent with $a_1\langle (1,1,1), v\rangle = 3a_1$, that is, with $\langle (1,1,1), v\rangle = 3$. Hence the plane V equals

$$V = \{v \in \mathbb{R}^3 \ : \ \langle (1,1,1), v\rangle = 3\}.$$

This plane does indeed contain p_1, p_2, and p_3 (do not forget to check this), so there is indeed a unique plane containing p_1, p_2, p_3. We may choose $a = (1,1,1)$ and $b = 3$.

1.5.2 With $\lambda = \frac{\langle a,v\rangle}{\langle a,a\rangle} = \frac{13}{5}$ we get $v_1 = \lambda a = \frac{1}{5}(26, 13)$ and $v_2 = v - v_1 = \frac{1}{5}(-6, 12)$.

1.5.3 With $\lambda = \frac{\langle a,v\rangle}{\langle a,a\rangle} = \frac{2x_1+x_2}{5}$ we get $v_1 = \lambda a = \frac{1}{5}(4x_1 + 2x_2, 2x_1 + x_2)$ and $v_2 = v - v_1 = \frac{1}{5}(x_1 - 2x_2, -2x_1 + 4x_2)$.

1.5.6 We hebben $a \in F^n$, met $a \neq 0$. Volgens Propositie 1.28 en Definitie 1.29 hebben we dan, voor alle $x \in F^n$:

$$\pi_{L(a)}(x) = \frac{\langle x, a\rangle}{\langle a, a\rangle} \cdot a.$$

Dus geldt voor alle x en y in F^n:

$$\begin{aligned}\pi_{L(a)}(x+y) &= \frac{\langle x+y,a\rangle}{\langle a,a\rangle}\cdot a = \left(\frac{\langle x,a\rangle + \langle y,a\rangle}{\langle a,a\rangle}\right)\cdot a \\ &= \left(\frac{\langle x,a\rangle}{\langle a,a\rangle} + \frac{\langle y,a\rangle}{\langle a,a\rangle}\right)\cdot a = \frac{\langle x,a\rangle}{\langle a,a\rangle}\cdot a + \frac{\langle y,a\rangle}{\langle a,a\rangle}\cdot a \\ &= \pi_{L(a)}(x) + \pi_{L(a)}(y).\end{aligned}$$

Voor alle $\lambda \in F$ en alle $x \in F^n$ geldt:

$$\pi_{L(a)}(\lambda\cdot x) = \frac{\langle \lambda\cdot x,a\rangle}{\langle a,a\rangle}\cdot a = \frac{\lambda\langle x,a\rangle}{\langle a,a\rangle}\cdot a = \lambda\cdot\frac{\langle x,a\rangle}{\langle a,a\rangle}\cdot a = \lambda\cdot\pi_{L(a)}(x).$$

Voor alle $x \in F^n$ geldt $x = \pi_{L(a)}(x) + \pi_{a^\perp}(x)$, dus ook $\pi_{a^\perp}(x) = x - \pi_{L(a)}(x)$. Daarom geldt voor alle x en y in F^n:

$$\begin{aligned}\pi_{a^\perp}(x+y) &= x+y-\pi_{L(a)}(x+y) = x+y-(\pi_{L(a)}(x)+\pi_{L(a)}(y)) \\ &= x+y-\pi_{L(a)}(x)-\pi_{L(a)}(y) = x-\pi_{L(a)}(x)+y-\pi_{L(a)}(y) \\ &= \pi_{a^\perp}(x)+\pi_{a^\perp}(y).\end{aligned}$$

Voor alle $\lambda \in F$ en alle $x \in F^n$ geldt:

$$\begin{aligned}\pi_{a^\perp}(\lambda\cdot x) &= \lambda\cdot x - \pi_{L(a)}(\lambda\cdot x) = \lambda\cdot x - \lambda\cdot\pi_{L(a)}(x) \\ &= \lambda\cdot(x-\pi_{L(a)}(x)) = \lambda\cdot\pi_{a^\perp}(x).\end{aligned}$$

1.6.1 See 1.5.2: We have $v_1 = \frac{13}{5}(2,1)$ and $v_2 = \frac{6}{5}(-1,2)$, so

$$\begin{aligned}d(p,L(a)) &= \|v_2\| = \tfrac{6}{5}\sqrt{(-1)^2+2^2} = \tfrac{6}{5}\sqrt{5} \text{ and} \\ d(p,V) &= \|v_1\| = \tfrac{13}{5}\sqrt{2^2+1^2} = \tfrac{13}{5}\sqrt{5}.\end{aligned}$$

1.6.3 Laat $x = (1,1,1,1)$ en $a = (1,2,3,4)$. Die afstand is

$$\|x-\pi_{L(a)}(x)\| = \|x - \frac{\langle x,a\rangle}{\langle a,a\rangle}\cdot a\| = \|x - \frac{1}{3}a\| = \|(1/3)\cdot(2,1,0,-1)\| = \frac{\sqrt{6}}{3}.$$

1.6.4 We translate the line L over $-p$ to obtain the line

$$L' = L - p = \{x - p : x \in L\} = L(w).$$

We also translate v and obtain $v' = v - p = (1,-1,0)$. The distance from v to L is the same as the distance from v' to L', so we get

$$d(v,L) = d(v',L') = d(v',L(w)) = \|\pi_{w^\perp}(v')\|.$$

With $\lambda = \frac{\langle w,v'\rangle}{\langle w,w\rangle} = \frac{1}{30}$ we find

$$\pi_w(v') = \tfrac{1}{30}w \qquad \text{and} \qquad \pi_{w^\perp}(v') = v' - \pi_w(v') = \tfrac{1}{30}(28,-31,-5),$$

so the distance is $\frac{1}{30}\sqrt{28^2+(-31)^2+)-5)^2} = \frac{1}{30}\sqrt{1770}$.

1.6.7 Laat $a = (1,1,1)$ en $x = (x_1,x_2,x_3)$ in $\mathbb{R}^3$. Dan hebben we:

$$d(x,a^\perp) = \|\pi_{L(a)}(x)\| = \frac{|\langle x,a\rangle|}{\|a\|} = \frac{|x_1+x_2+x_3|}{\sqrt{3}}.$$

En ook:

$$\begin{aligned}d(x,L(a))^2 &= \|x\|^2 - \|\pi_{L(a)}(x)\|^2 \\ &= x_1^2+x_2^2+x_3^2-(x_1+x_2+x_3)^2/3,\end{aligned}$$

dus $d(x, L(a)) = \sqrt{x_1^2 + x_2^2 + x_3^2 - (x_1 + x_2 + x_3)^2/3}$.

1.7.1 Voor $a = (1,2) \in L$ geldt $L = L(a)$. We schrijven $M = a^\perp$, de lijn door 0 die loodrecht staat op L. De projectie van $v = (5,0)$ op de lijn L is $\pi_L(v) = \lambda a$ met $\lambda = \frac{\langle a,v \rangle}{\langle a,a \rangle} = \frac{5}{5} = 1$, dus $\pi_L(v) = a = (1,2)$. (Inderdaad staat $v - \lambda a = v - a = (4,-2)$ loodrecht op a.) De spiegeling van v in L is dan $s_L(v) = 2\pi_L(v) - v = (2,4) - (5,0) = (-3,4)$.

(Inderdaad kunnen we nu checken dat $\tilde{v} = (-3,4)$ voldoet aan de eisen van 1.49: de vector $v - \tilde{v} = (8,-4)$ is een normaal van L en $\frac{1}{2}(v + \tilde{v}) = (1,2) = a \in L$.)

1.7.3 We have $V = a^\perp$. Set $v = (1,2,-1)$ and $\lambda = \frac{\langle a,v \rangle}{\langle a,a \rangle} = -\frac{1}{25}$. Then we have $\pi_a(v) = \lambda a$ and $\pi_V(v) = v - \pi_a(v) = \frac{1}{25}(28, 50, -21)$. We have $v = v_1 + v_2$ with $v_1 = \pi_a(v)$ and $v_2 = \pi_V(v)$. We compute

$$s_a(v) = v_1 - v_2 = \tfrac{1}{25}(-31, -50, 17),$$
$$s_V(v) = -v_1 + v_2 = \tfrac{1}{25}(31, 50, -17).$$

1.7.4 We have $s_W(v) = 2\pi_W(v) - v$, so for $x, y \in F^n$ we find from Exercise 1.5.6 that

$$\begin{aligned} s_W(x+y) &= 2\pi_W(x+y) - (x+y) = 2(\pi_W(x) + \pi_W(y)) - (x+y) \\ &= (2\pi_W(x) - x) + (2\pi_W(y) - y) = s_W(x) + s_W(y) \end{aligned}$$

and for each $\lambda \in F$ we get

$$s_W(\lambda x) = 2\pi_W(\lambda x) - (\lambda x) = 2\lambda\pi_W(x) - (\lambda x) = \lambda(2\pi(x) - x) = \lambda s_W(x).$$