Uitwerkingen werkcollege 5.

3.5.1 From $S \subset S \cup T$ and $T \subset S \cup T$ we find $L(S) \subset L(S \cup T)$ and $L(T) \subset L(S \cup T)$ (part (1) of Proposition 3.33), so $L(S) \cup L(T) \subset L(S \cup T)$. Since $L(S \cup T)$ is a linear subspace, we then find from Lemma 3.29, applied to $L(S) \cup L(T)$, that

$$L(S) + L(T) = L\big(L(S) \cup L(T)\big) \subset L(S \cup T).$$

For the converse we use that $S \subset L(S)$ and $T \subset L(T)$, so

$$(S \cup T) \subset (L(S) \cup L(T)) \subset L\big(L(S) \cup L(T)\big) = L(S) + L(T).$$

Since $L(S)+L(T)$ is a linear subspace, we then obtain $L(S \cup T) \subset L(S)+L(T)$ from Lemma 3.29. Together these inclusions prove $L(S \cup T) = L(S)+L(T)$.

3.5.4 If U_1 and U_2 are contained in U, then so is the union $U_1 \cup U_2$ and by Lemma 3.29, so is $L(U_1 \cup U_2) = U_1 + U_2$.

3.5.7 No, because the intersection is not equal to $\{0\}$, because, for example, the function f given by $f(x) = x^2 - x$ is contained in $U_1 \cap U_2$.

4.1.4 The first is not linear because 0 (that is, $(0,0,0)$) does not map to 0 (that is, $(0,0)$). The second is not linear, because if we call the map f, then we have $f(2e_1) = (4,0,0)$ and $f(e_1) = (1,0,0)$, so $f(2e_1) \neq 2f(e_1)$, so f does not respect scalar multiplication. The others, so (3), (4), (5), and (6), are linear.

4.1.7 (1) Let $x, y \in \mathbb{R}^2$ be two points. Then the points $0, x, x+y, y$ are the vertices of a parallelogram, with the points lying in that order around the perimeter. Since the rotation ρ preserves angles, the images $\rho(0) = 0$ and $\rho(x), \rho(y), \rho(x+y)$ also form a parallelogram, in that order around the perimeter. This means that we have $\rho(x+y) = \rho(x) + \rho(y)$. Because ρ also preserves distances, we also have $\rho(\lambda x) = \lambda\rho(x)$, so ρ is a linear map.

(2) The image of $(1,0)$ is $(\cos\theta, \sin\theta)$ and the image of $(0,1)$ is $(-\sin\theta, \cos\theta)$.

(3) Since ρ is linear, it sends $(x,y) = x(1,0) + y(0,1)$ to (use part (2))

$$\begin{aligned} x\rho((1,0)) + y\rho((0,1)) &= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

4.2.2 (1) By Example 4.21 we have $\rho^2 + \rho = -\operatorname{id}$. Therefore, Proposition 4.26 yields

$$\begin{aligned} f \circ g &= (\rho - \operatorname{id}) \circ (\rho + 2\operatorname{id}) \\ &= \rho \circ \rho - \operatorname{id} \circ \rho + \rho \circ (2\operatorname{id}) - \operatorname{id} \circ (2\operatorname{id}) \\ &= \rho^2 - \rho + 2\rho - 2\operatorname{id} \\ &= (\rho^2 + \rho) - 2\operatorname{id} = -\operatorname{id} - 2\operatorname{id} = -3\operatorname{id}. \end{aligned}$$

We similarly obtain $g \circ f = -3\operatorname{id}$.

(2) From part (1) we obtain $f \circ (-\frac{1}{3}g) = (-\frac{1}{3}g) \circ f = \operatorname{id}$, so $(-\frac{1}{3}g)$ is the inverse of f, so f is an isomorphism. Similarly, $(-\frac{1}{3}f)$ is an inverse of g, so g is an isomorphism as well.

4.2.5 (0) We moeten bewijzen dat π_U lineair is. Laat v_1 en v_2 in V zijn, en λ in F. Dan zijn er unieke $u_1 \in U$ en $u_1' \in U'$ zodat $v_1 = u_1 + u_1'$. Ook zijn er unieke $u_2 \in U$ en $u_2' \in U'$ zodat $v_2 = u_2 + u_2'$. Per definitie gelden $\pi_U(v_1) = u_1$ en $\pi_U(v_2) = u_2$. We hebben:

$$v_1 + v_2 = (u_1 + u_1') + (u_2 + u_2') = (u_1 + u_2) + (u_1' + u_2'),$$

en omdat U en U' deelruimten zijn hebben we $u_1+u_2 \in U$ en $u_1'+u_2' \in U'$, dus dit zijn de unieke elementen van U en U' waarvan de som $v_1 + v_2$ is. Er geldt dus $\pi_U(v_1 + v_2) = u_1 + u_2 = \pi_U(v_1) + \pi_U(v_2)$.

(1) De definitie van π_U geeft $\mathrm{im}(\pi_U) \subseteq U$. We gaan de andere inclusie bewijzen. Laat $u \in U$. Dan $\pi_U(u) = u$, want $u = u + 0$ en $u \in U$ en $0 \in U'$. Nu bewijzen we $\ker(\pi_U) = U'$ door de twee inclusies te bewijzen. Laat $v \in U'$. Dan $v = 0 + v$ met $0 \in U$ en $v \in U'$ dus $\pi_U(v) = 0$ en $v \in \ker(\pi_U)$. Stel nu $v \in \ker(\pi_U)$. Er unieke $u \in U$ en $u' \in U'$ met $v = u + u'$. Dan $0 = \pi_U(v) = u$, dus $v = 0 + u' \in U'$.

(2) Laat $v \in V$, en $u \in U$ en $u' \in U'$ de unieke elementen zodat $v = u+u'$. Dan $\pi_U(v) = u$, en omdat $u = u+0$ met $u \in U$ en $0 \in U'$, $\pi_U(u) = u$. Dus voor alle v in V hebben we $\pi_U(\pi_U(v)) = \pi_U(u) = u$, dus $\pi_U \circ \pi_U = \pi_U$.

(3) Laat $v \in V$. Laat $u \in U$ en $u' \in U'$ de unieke elementen zijn met $v = u + u'$. Dan $(\mathrm{id}_V - \pi_U)(v) = \mathrm{id}_V(v) - \pi_U(v) = v - u = u'$. Aan de andere kant geldt $\pi_{U'}(v) = u'$. Dus voor alle $v \in V$ geldt dat $(\mathrm{id}_V - \pi_U)(v) = \pi_{U'}(v)$, dus $\mathrm{id}_V - \pi_U = \pi_{U'}$.

4.2.6 (1) Laat $v \in V$. We berekenen:

$$\begin{aligned}\pi(v - \pi(v)) &= \pi(v) - \pi(\pi(v)) = \pi(v) - (\pi \circ \pi)(v) \\ &= \pi(v) - \pi(v) = 0.\end{aligned}$$

Dus $v - \pi(v)$ is in $\ker(\pi) = U'$.

(2) We laten zien dat $U \cap U' = \{0\}$. Laat $v \in U \cap U'$. Dan $v \in U'$, dus $\pi(v) = 0$ Ook: $v \in U$, dus er is een $w \in V$ met $v = \pi(w)$. Dan geldt $0 = \pi(v) = \pi(\pi(w)) = \pi(w) = v$. Nu laten we zien dat $U + U' = V$. Laat $v \in V$, dan $v = \pi(v) + (v - \pi(v))$, met $\pi(v) \in U$ en $v - \pi(v) \in U'$ vanwege (1).

(3) Voor elke v in V hebben we $v = \pi(v) + (v - \pi(v))$, met $\pi(v) \in U$ en $v - \pi(v) \in U'$, en deze ontbinding is uniek omdat U en U' complementair zijn in V, dus geldt $\pi(v) = \pi_U(v)$. Dit geldt voor alle v, dus $\pi = \pi_U$.

4.3.1

4.1.4.(3) $n = 3$, $W = \mathbb{C}^4$, $C = \big((1, 1, 0, 1), (2, 0, 1, 2), (0, -3, -1, 1)\big)$.

4.1.4.(4) $n =$, $W = V$, $C = \big(v_1, v_2, v_3\big)$.

4.1.7 $n = 2$, $W = \mathbb{R}^2$, $C = \big((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\big)$.

4.1.8 $n = 2$, $W = \mathbb{R}^2$, $C = \big((0, -1), (-1, 0)\big)$.

4.1.10 $n = 2$, $W = \mathbb{R}^2$, $C = \big((\frac{3}{5}, \frac{4}{5}), (\frac{4}{5}, -\frac{3}{5})\big)$.

4.1.11 $n = 2$, $W = \mathbb{R}^2$, $C = \big((-\frac{4}{5}, \frac{3}{5}), (\frac{3}{5}, \frac{4}{5})\big)$.

4.3.2 (1) We take $m = n$, $W = F$, and $C = (0, 0, \dots, 0, 1, 0, \dots, 0) \in F^n$, with the 1 on the j-th position.

(2) The same: $a = (0, 0, \dots, 0, 1, 0, \dots, 0) \in F^n$, with the 1 on the j-th position!