Uitwerkingen werkcollege 7.

3.4.10 In all three cases it is clear that $v_i' \in L(v_1, v_2, \ldots, v_n)$ for all $i \in \{1, \ldots, n\}$. For each of the three cases we now show that we also have $v_i \in L(v_1', v_2', \ldots, v_n')$ for all $i \in \{1, \ldots, n\}$. Indeed, in case (1), this follows from the fact that we have $v_j = \lambda^{-1} v_j'$. In case (2), we have $v_k = v_k' - \lambda v_j'$. In case (3), we have $v_j = v_k'$ and $v_k = v_j'$. Hence, it follows from Lemma 3.32, applied to $S = \{v_1, \ldots, v_n\}$ and $T = \{v_1', \ldots, v_n'\}$, that we have

$$L(v_1', v_2', \ldots, v_n') = L(v_1, v_2, \ldots, v_n) = W.$$

5.6.2 Identify x with an $n \times 1$matrix and y with an $m \times 1$ matrix. Then as in Remark 5.33 we have

$$\langle Mx, y\rangle = (Mx)^\top \cdot y = (x^\top M^\top) \cdot y = x^\top \cdot (M^\top y) = \langle x, M^\top y\rangle.$$

5.6.3

$$a^\top \cdot b = (24), \qquad a \cdot (b^\top) = \begin{pmatrix} -2 & 1 & 4 & 3 \\ -4 & 2 & 8 & 6 \\ -6 & 3 & 12 & 9 \\ -8 & 4 & 16 & 12 \end{pmatrix}.$$

6.1.3

$$B_1 = N_{1,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B_2 = M_{2,1}(-2) \cdot M_{4,1}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$B_3 = M_{3,2}(4) \cdot M_{4,2}(5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{pmatrix} \qquad B_4 = M_{4,3}(-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$B_5 = M_{3,4}(-3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B_6 = M_{4,3}(-2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$$B_7 = N_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad B_8 = M_{4,3}(-4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix}$$

$$B_9 = L_4(-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$B = B_9 \cdot B_8 \cdot B_7 \cdot B_6 \cdot B_5 \cdot B_4 \cdot B_3 \cdot B_2 \cdot B_1.$$

6.2.1 These answers are not unique (though it should be relatively easy to check any other answer: they should be in row echelon form as well, and they should have the same row space).

$$A_1' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2' = \begin{pmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 20 \end{pmatrix}$$

$$A_3' = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_4' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_5' = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

6.3.3

$$\begin{pmatrix} 1 & 3 & 6+2i \\ 0 & 1 & \frac{9}{17}(4+i) \end{pmatrix} \qquad \text{with kernel generated by } w = \begin{pmatrix} \frac{1}{17}(6-7i) \\ -\frac{9}{17}(4+i) \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{with kernel generated by } w = \begin{pmatrix} -1 \\ \frac{2}{3} \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ with kernel generated by } w_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ en } w_5 = \begin{pmatrix} 2 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with kernel generated by } w_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

6.3.4 (1) If f_A is injective, then the kernel of A is trivial, that is, $\ker A = \{0\}$. Therefore, every column in a row echelon form A' for A contains a pivot. This means there are n pivots, and as each of the m rows contains at most one pivot, there are at least n rows, so $m \geq n$.

(2) If A is invertible, then f_A is an isomorphism, so both f_A and its inverse $f_{A^{-1}}$ are injective. Applying part (1) to both A and A^{-1} we find both $m \geq n$ and $n \geq m$, so $m = n$.

6.3.5 Note that since f_A is linear, the hyperplane H contains 0. We answer this question in two different ways.

(1) The projection $\pi_H \colon \mathbb{R}^4 \to \mathbb{R}^4$ and the reflection $s_H = f_A \colon \mathbb{R}^4 \to \mathbb{R}^4$ are related by $s_H = 2\pi_H - \mathrm{id}$ by Example 4.22, or, equivalently, $2\pi_H = s_H + \mathrm{id} = f_A + \mathrm{id} = f_A + f_I = f_{A+I}$. The image of π_H (and therefore of of $2\pi_H$) is therefore equal to the image of f_{A+I}, which is the column space of

$$A + I = \frac{1}{7} \cdot \begin{pmatrix} 12 & -4 & -2 & 2 \\ -4 & 6 & -4 & 4 \\ -2 & -4 & 12 & 2 \\ 2 & 4 & 2 & 12 \end{pmatrix}.$$

This is one way to answer the question, as it does not ask for a specific way to represent it. We could now also find a normal of H by computing the kernel of $A+I$. A row echelon form for $7(A+I)$ is

$$\begin{pmatrix} 1 & 2 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the kernel is generated by

$$a = \begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \end{pmatrix},$$

which means that we have $H = a^\perp$.

(2) Set $L = H^\perp$, which is a line through 0 The projection $\pi_L \colon \mathbb{R}^4 \to \mathbb{R}^4$ and the reflection $s_H = f_A \colon \mathbb{R}^4 \to \mathbb{R}^4$ are related by $s_H = \mathrm{id} - 2\pi_L$ by Example 4.22, or, equivalently, $2\pi_L = \mathrm{id} - s_H = \mathrm{id} - f_A = f_{I-A}$. The image of π_L (and therefore of $2\pi_L$) is therefore equal to the image of f_{I-A}, which is the column space of

$$I - A = \frac{1}{7} \cdot \begin{pmatrix} 2 & 4 & 2 & -2 \\ 4 & 8 & 4 & -4 \\ 2 & 4 & 2 & -2 \\ -2 & -4 & -2 & 2 \end{pmatrix}.$$

The columns of $I - A$ are all multiples of the vector

$$b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix},$$

so L is generated by b and we have $H = b^\perp$.