Uitwerkingen werkcollege 9.

Voorbeeld 6.16 De opgave is om voortbrengers van de kern te vinden voor de matrix A uit voorbeeld 6.16 via de gereduceerde rijtrapvorm A' van A. Je krijgt de gereduceerde rijtrapvorm door:
(1) rij 3 -2 keer bij rijen 1 en 2 op te tellen,
(2) rij 2 1 keer bij rij 1 op te tellen.
Dat geeft

$$A' = \begin{pmatrix} \textcircled{1} & 2 & 0 & -1 & 0 & -4 & -5 \\ 0 & 0 & \textcircled{1} & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

De vrije variabelen zijn x_2, x_4, x_6 en x_7. De oplossing met $(x_2, x_4, x_6, x_7) = (1, 0, 0, 0)$ is $w_1 = (-2, 1, 0, 0, 0, 0, 0)$. De oplossing met $(x_2, x_4, x_6, x_7) = (0, 1, 0, 0)$ is $w_2 = (1, 0, 1, 1, 0, 0, 0)$. De oplossing met $(x_2, x_4, x_6, x_7) = (0, 0, 1, 0)$ is $w_3 = (4, 0, 3, 0, -1, 1, 0)$. De oplossing met $(x_2, x_4, x_6, x_7) = (0, 0, 0, 1)$ is $w_4 = (5, 0, 0, 0, -1, 0, 1)$. Merk op dat de coördinaten x_1, x_3 en x_5 van w_i rechtstreeks uit de kolom van A' komen die hoort bij de vrije variabele die ongelijk nul is, maar met een min-teken.

7.1.1 (1) Yes. (2) No.

7.1.4 Let A be the 5×6 matrix with $v_1, \ldots, v_6$ as columns. Then there are more columns than rows, so not every column of a row echelon form for A contains a pivot, so the kernel of f_A is nontrivial, so the elements $v_1, \ldots, v_6$ are linearly dependent by Proposition 7.10 and/or Corollary 7.11. See also Exercise 6.3.4.

7.1.8 Suppose $0 \leq j \leq n$. For all $0 \leq i < j$ we have $f_i(a_j) = 0$, so for every linear combination f of $f_0, f_1, \ldots, f_{j-1}$ we have $f(a_j) = 0$. The function f_j satisfies $f_j(a_j) = 1$, so it is not a linear combination of $f_0, f_1, \ldots, f_{j-1}$. This holds for all j, so by part (1) of Proposition 7.15, the sequence $(f_0, f_1, \ldots, f_n)$ is indeed linearly independent.

7.2.1 Many answers are possible in this case. While one could use Proposition 7.22, we will use Proposition 7.26 (or Lemma 7.24). We don't list the matrices and their row echelon forms here, but only the conclusion.
(1) (v_1, v_2), which generates $\mathbb{R}^2$.
(2) (v_1, v_2), note that $5v_3 = v_1 + 2v_2$, so $v_3 \in L(v_1, v_2)$.
(3) (v_1, v_2), note that $4v_3 = v_1 + v_2$, so $v_3 \in L(v_1, v_2)$.
(4) (v_1, v_2), note that $v_3 = -v_1 + 2v_2 \in L(v_1, v_2)$.
(5) (v_1, v_2, v_3).

7.2.3 We select the columns of A corresponding to the columns of a row echelon form of A that contain a pivot (see Proposition 7.26). In Exercise 6.3.3 we have seen row echelon forms for each of the four matrices, so we can choose:
(1) for the first matrix: the first two columns,

(2) for the second (the 3×3 matrix): the first two columns,
(3) for the third (the 3×5 matrix): the first, second, and fourth column,
(4) for the last matrix: the first, second, and fourth column.

2e huiswerkopgave Gegeven is de matrix A in $\mathrm{Mat}(2 \times 4, \mathbb{R})$ met rijen $(0,0,2,3)$ en $(-1,0,4,5)$ en gevraagd is de gereduceerde rijtrapvorm te bepalen met het algoritme van Propositie 6.9 en de uitbreiding van stap 5 onderdaan p. 112. We starten met

$$A = \begin{pmatrix} 0 & 0 & 2 & 3 \\ -1 & 0 & 4 & 5 \end{pmatrix}.$$

We reduceren de volgende matrix naar gereduceerde rijtrapvorm:

$$\begin{pmatrix} 0 & 0 & 2 & 3 \\ -1 & 0 & 4 & 5 \end{pmatrix} \rightsquigarrow \begin{matrix} R_2 \\ R_1 \end{matrix} \begin{pmatrix} -1 & 0 & 4 & 5 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$
$$\rightsquigarrow \begin{matrix} -1{\cdot}R_1 \\ R_2 \end{matrix} \begin{pmatrix} 1 & 0 & -4 & -5 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$
$$\rightsquigarrow \begin{matrix} R_1 \\ R_2/2 \end{matrix} \begin{pmatrix} 1 & 0 & -4 & -5 \\ 0 & 0 & 1 & 3/2 \end{pmatrix}$$
$$\rightsquigarrow \begin{matrix} R_1 + 4R_2 \\ R_2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3/2 \end{pmatrix}.$$

De vrije variabelen zijn x_2 en x_4. Daarbij hoort de basis w_1, w_2 van de kern gegeven door voor w_1 $(x_2, x_4) = (1,0)$ te nemen, dat geeft $w_1 = (0,1,0,0)$, en met $(x_2, x_4) = (0,1)$ krijgen we $w_2 = (-1,0,-3/2,1)$.

7.3.1 As in Example 7.31, we look at the matrix whose columns are the coefficients of the polynomials f_1, f_2, f_3 and the generators $1, x, x^2, x^3, x^4$, which is

$$A = \begin{pmatrix} 2 & -3 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A row echelon form of this matrix is

$$A' = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{1}{7} & 0 & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The first three columns of A' contain a pivot, so they are indeed linearly independent. The other two pivots, in the fifth and eighth column, correspond to x and x^4, so we can extend to a basis f_1, f_2, f_3, x, x^4.

7.3.3 (1) The hyperplane V is equal to the kernel of the 1×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix},$$

which is in row echelon form. By Proposition 7.22, a basis for this kernel is given by the elements obtained from Proposition 6.17, which are

$$w_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad w_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the dimension of V is 3, corresponding to the number of columns of A without a pivot.

(2)&(3) We have $\langle a, v_1 \rangle = \langle a, v_2 \rangle = 0$, so $v_1, v_2 \in a^\perp = V$. The matrix that has v_1, v_2, w_2, w_3, w_4 as columns is

$$B = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -3 & 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the last three columns generate V by part (1). A row echelon form for B is

$$B' = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first two columns contain a pivot, so v_1 and v_2 are indeed linearly independent (which could also be seen very quickly, as it is just two vectors and neither is a scalar multiple of the other). The only other column with a pivot is the fourth, which corresponds with w_3, so we can extend (v_1, v_2) to a basis (v_1, v_2, w_3) of V.