**Uitwerkingen werkcollege 11.**

8.4.1

$$\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ -5 & -4 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -3 & -2 & -2 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 & -1 & 2 \\ -9 & 1 & 3 & -8 \\ 8 & -1 & -3 & 7 \\ -8 & 1 & 3 & -8 \end{pmatrix}.$$

8.4.4 We use the definition in Definition 5.22: $AB$ is invertible means that $f_{AB}$ is an isomorphism. Then $f_{AB} = f_A \circ f_B \colon F^n \to F^n$ is a bijection. Its injectivity implies that $f_B \colon F^n \to F^n$ is injective and its surjectivity implies that that $f_A$ is surjective. From Corollary 8.5 we then conclude that $f_A$ and $f_B$ are isomorphisms as well, so $A$ and $B$ are invertible as well.

8.4.5 Since $I_n$ is invertible, it follows from Exercise 8.4.4 that $M$ and $N$ are invertible. Note that from $MN = I_n$, we obtain $f_M \circ f_N = \mathrm{id}$, so $f_N$ is a right inverse of the isomorphism $f_M$. Since every right inverse of a bijection is equal to the inverse of that bijection (see Appendix A), we find that $f_N$ is the inverse of $f_M$ and hence $N = M^{-1}$. It follows that we also have $NM = M^{-1}M = I_n$.

Alternatively, we can use Lemma 8.30. Parts (6) and (7) give that $M$ and $N$ are invertible, and the last statement in Lemma 8.30 says that and $M = N^{-1}$ and $N = M^{-1}$. Then $NM = M^{-1}M = I_n$.

8.5.1 For the first three systems we have

$$A = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 2 & 2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \text{with} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

respectively. We can reduce the work by extending the matrix $A$ with all three choices for $b$ and finding a reduced row echelon form. The extended matrix is

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 1 & 1 \\ 3 & 2 & 2 & 0 & -1 & 1 \\ 0 & -1 & 2 & 0 & -1 & 1 \end{array}\right).$$

and it has reduced row echelon form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Hence the kernel of $A$ is generated by $a = (-2, 2, 1)$, that is $\ker A = L(a)$. Now for the first $b$, namely $b = 0$, the solution set is $\ker A = L(a)$. For the second $b$, if we had extended $A$ by only $b$, the last column of reduced row echelon form of this extension $(A|b)$ does not have a pivot in the last column, so the system is consistent. We obtain a solution by setting $x = (x_1, x_2, 1)$ and solving for $x_1$ and $x_2$, which gives $x = (-3, 3, 1)$. Therefore, the complete solution space is

$$\{(-3, 3, 1) + z \ : \ z \in \ker A\} = \{(-3, 3, 1) + \lambda a \ : \ \lambda \in \mathbb{R}\}.$$

For the third $b$, namely $b = (1,1,1)$, the reduced row echelon form of the extended matrix $(A|b)$ has a pivot in the last column, so there is no solution. For the last case, we have

$$A = \begin{pmatrix} 3 & 1 & 2 & -2 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

The extended matrix $(A|b)$ has reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

Hence, there is a unique solution, namely $x = (-13, 4, 16, -2)$.

9.1.2 We berekenen de beelden van de elementen van de begin-basis $B$ en schrijven die uit als lineaire combinaties van de elementen van de eind-basis $B$.

$$\begin{aligned} T(1) &= 3 = 3{\cdot}1 + 0{\cdot}x + 0{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\ T(x) &= 3x = 0{\cdot}1 + 3{\cdot}x + 0{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\ T(x^2) &= 3x^2 + (x-2){\cdot}2 = -4{\cdot}1 + 2{\cdot}x + 3{\cdot}x^2 + 0{\cdot}x^3 + 0{\cdot}x^4 \\ T(x^3) &= 3x^3 + (x-2){\cdot}6x = 0{\cdot}1 - 12{\cdot}x + 6{\cdot}x^2 + 3{\cdot}x^3 + 0{\cdot}x^4 \\ T(x^4) &= 3x^4 + (x-2){\cdot}12x^2 = 0{\cdot}1 + 0{\cdot}x - 24{\cdot}x^2 + 12{\cdot}x^3 + 3{\cdot}x^4 \end{aligned}$$

Dus

$$[T]_B^B = \begin{pmatrix} 3 & 0 & -4 & 0 & 0 \\ 0 & 3 & 2 & -12 & 0 \\ 0 & 0 & 3 & 6 & -24 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

9.1.3 The map $T\colon F[x]_{k-1} \to F^k$ that sends $g \in F[x]_{k-1}$ to $(g(\alpha_1), g(\alpha_2), \dots, g(\alpha_k))$ is an isomorphism by Exercise 8.1.2 (with $n = k-1$). By Example 9.5, the matrix, say $A$, associated to $T$ with respect to the basis $B = (1, x, \dots, x^{k-1})$ of $F[x]_{k-1}$ and the standard basis $E$ of $F^k$ is exactly the given Vandermonde matrix. Therefore, the map $f_A\colon F^k \to F^k$ is equal to the composition $\varphi_E^{-1} \circ T \circ \varphi_B = T \circ \varphi_B$ of isomorphisms, so it is an isomorphism, and therefore $A$ is invertible.

9.1.4 We geven alleen de uitkomst:

$$\begin{pmatrix} 3 & 0 & 7 & 0 \\ 0 & 3 & 0 & 7 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 5 \\ 8 & 0 & 2 & 0 \\ 0 & 8 & 0 & 2 \end{pmatrix}$$

9.2.1 (1)

$$M = [\mathrm{id}]_B^{B'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \text{and} \qquad N = M^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Of course, instead of finding the inverse of $M$, one can also find $N$ by noticing that $v_1 = v_1'$ and $iv_i = v_i' - v_{i-1}'$ for $2 \leq i \leq 4$.

(2) we have $f_M = \varphi_B^{-1} \circ \mathrm{id} \circ \varphi_{B'} = \varphi_B^{-1} \circ \varphi_{B'}$, so for every $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we find

$$Mx = f_M(x) = \varphi_B^{-1}(\varphi_{B'}(x)) = \varphi_B^{-1}(x_1 v_1' + \ldots + x_4 v_4') = (x_1 v_1' + \ldots + x_4 v_4')_{B'}.$$

(3) Same as (2) with the roles of $M$ and $N$ reversed (as well as the roles of $v_i$ and $v_i'$).

9.2.2

$$[\mathrm{id}]_E^C = \begin{pmatrix} -1 & -2 & 1 \\ -2 & 1 & -1 \\ 0 & 3 & -2 \end{pmatrix} \qquad \text{and} \qquad [\mathrm{id}]_C^E = ([\mathrm{id}]_E^C)^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -4 & 2 & -3 \\ -6 & 3 & -5 \end{pmatrix}.$$