Uitwerkingen werkcollege 4.

3.1.1 Clearly, the element $(0,0,0)$ satisfies the equation, so we have $(0,0,0) \in V$. Suppose we have $x = (x_1, x_2, x_3) \in V$ and $y = (y_1, y_2, y_3) \in V$. Then we have $x_1+2x_2-3x_3 = 0$ and $y_1+2y_2-3y_3 = 0$. If we write $x+y = (z_1, z_2, z_3)$, then we have $z_i = x_i + y_i$ for all $i \in \{1,2,3\}$, so

$$\begin{aligned} z_1 + 2z_2 - 3z_3 &= (x_1 + y_1) + 2(x_2 + y_2) - 3(x_3 + y_3) \\ &= (x_1 + 2x_2 - 3x_3) + (y_1 + 2y_2 - 3y_3) = 0 + 0 = 0, \end{aligned}$$

so $x + y \in V$.

Suppose $\lambda \in \mathbb{R}$. If we write $\lambda x = (w_1, w_2, w_3)$, then we have $w_i = \lambda x_i$ for $i \in \{1,2,3\}$, so

$$w_1 + 2w_2 - 3w_3 = \lambda x_1 + 2\lambda x_2 - 3\lambda x_3 = \lambda(x_1 + 2x_2 - 3x_3) = \lambda \cdot 0 = 0,$$

so $\lambda x \in V$. This proves that V is indeed a linear subspace.

3.1.2 No, because the zero element $(0,0,0)$ of $\mathbb{R}^3$ is not contained in U.

3.1.4 Since the set $\mathbb{R}[x]$ of all polynomials with the given addition and scalar multiplication is a vector space (Example 2.11), it suffices to show that $\mathbb{R}[x]_{\leq d}$ is a subspace and therefore itself a vector space (Lemma 3.2).

Two elements $f, g \in \mathbb{R}[x]_{\leq d}$ can be written as $f = \sum_{i=1}^{d} a_i x^i$ and $g = \sum_{i=1}^{d} b_i x^i$, so their sum is $f + g = \sum_{i=1}^{d} (a_i + b_i) x^i$, which has degree at most d as well. The scalar multiple $\lambda f = \sum_{i=1}^{d} \lambda a_i x^i$ also has degree at most d, so $f+g$ and λf are elements of $\mathbb{R}[x]_{\leq d}$ as well. Since 0 has degree $-\infty$, we also have $0 \in \mathbb{R}[x]_{\leq d}$.

3.1.6 No. The constant function $f = 1$ that sends every $x \in X$ to 1 is contained in U, but the scalar multiple $2f$ is not contained in U, so U is not closed under scalar multiplication.

3.2.2 Clearly we have $0 \in S^{\perp}$. For the proof that $S^{\perp}$ is closed under addition and scalar multiplication, see Lemma 1.25. That was stated only when F is contained in $\mathbb{R}$, but its proof works for all fields F. This shows that $S^{\perp}$ is a linear subspace.

3.3.2
(1) Only H_2 is a subspace.
(2) Suppose $v = (x_1, x_2, x_3)$ is contained in $H_1 \cap H_2 \cap H_3$. Then all three equations are satisfied. The first and third equation together yield

$$\begin{aligned} 1 = 3 - 2 &= \langle (1,1,1), v \rangle - \langle (1,0,1), v \rangle \\ &= \langle (1,1,1) - (1,0,1), v \rangle = \langle (0,1,0), v \rangle = x_2. \end{aligned}$$

The second and first equation then give

$$\begin{aligned} -2 = 0 - 2 &= \langle (-1,2,1), v \rangle - \langle (1,0,1), v \rangle = \langle (-1,2,1) - (1,0,1), v \rangle \\ &= \langle (-2,2,0), v \rangle = -2x_1 + 2x_2 = -2x_1 + 2, \end{aligned}$$

so $x_1 = 2$. The first equation then implies

$$2 = \langle (1,0,1), v \rangle = x_1 + x_3 = 2 + x_3,$$

so $x_3 = 0$ and $v = (2,1,0)$. This only shows that if v satisfies all three equations, then v equals $(2,1,0)$. But it is also easy to check that this vector does indeed satisfy all three equations, so we have

$$H_1 \cap H_2 \cap H_3 = \{(2,1,0)\}.$$

3.3.3 $V = \mathbb{R}$ (over the field $\mathbb{R}$), and $U_1 = \{0, 1\}$ and $U_2 = \{0, -1\}$. Then $U_1 \cap U_2 = \{0\}$ is a linear subspace of $\mathbb{R}$, but U_1 and U_2 are not.

3.4.1 Statement (2) is proved in the book, so we prove statements (1), (3), and (4). First statement (1). Suppose $T \subset S$. Take $x \in S^\perp$. Then for every $s \in S$ we have $\langle x, s\rangle = 0$, so certainly for all $t \in T$ we have $\langle x, t\rangle = 0$. Therefore, we have $x \in T^\perp$, and we obtain $S^\perp \subset T^\perp$.

For (3), we first prove $S \subset (S^\perp)^\perp$. Suppose $s_0 \in S$. For all $x \in S^\perp$, we have $\langle x, s\rangle = 0$ for all $s \in S$, so in particular $\langle x, s_0\rangle = 0$. This implies $s_0 \in (S^\perp)^\perp$, so we conclude $S \subset (S^\perp)^\perp$. Since $(S^\perp)^\perp$ is a linear subspace by Proposition 3.20, we find $L(S) \subset (S^\perp)^\perp$ from Lemma 3.29.

For (4), suppose $T \subset F^n$ is a subset. For any element $x \in F^n$ we have the equivalences

$$\begin{aligned} x \in S^\perp \cap T^\perp &\Leftrightarrow x \in S^\perp \text{ and } x \in T^\perp \\ &\Leftrightarrow \langle x, s\rangle = 0 \text{ for all } s \in S \text{ and } \langle x, t\rangle = 0 \text{ for all } t \in T \\ &\Leftrightarrow \langle x, r\rangle = 0 \text{ for all } r \in S \cup T \\ &\Leftrightarrow x \in (S \cup T)^\perp. \end{aligned}$$

We conclude $S^\perp \cap T^\perp = (S \cup T)^\perp$.

3.4.6 No. Take $V = \mathbb{R}^2$ and $v = (1, 0)$. Take $I = \{v\}$ and $J = \{2v\}$. Then $I \cap J = \emptyset$, so $L(I \cap J) = \{0\}$. But $L(I) = L(J) = L(v)$, so we have $L(I) \cap L(J) = L(v)$.

3.4.7 (1) We gaan de drie eisen voor deelruimtes na. Laat $V^+ \subset \mathbb{R}^\mathbb{R}$ de deelverzameling van even functies zijn (zie Definitie 2.7 voor de betekenis van de notatie $\mathbb{R}^\mathbb{R}$). De nulfunctie is even, dus $0 \in V^+$. Gesloten onder optelling. Laat f en g in V^+, dan geldt, voor alle $x \in \mathbb{R}$:

$$\begin{aligned} (f+g)(-x) &= f(-x) + g(-x) = f(x) + g(x) \\ &= (f+g)(x), \end{aligned}$$

waarbij we achtereenvolgens hebben gebruikt: de definitie van $f + g$, dat f en g even zijn, en weer de definitie van $f + g$. Dus $f + g$ zit in V^+. Gesloten onder scalairvermenigvuldiging. Laat $\lambda \in \mathbb{R}$ en $f \in V^+$. Dan geldt, voor alle $x \in \mathbb{R}$:

$$(\lambda \cdot f)(-x) = \lambda \cdot (f(-x)) = \lambda \cdot (f(x)) = (\lambda \cdot f)(x),$$

waarbij we achtereenvolgens hebben gebruikt: de definitie van $\lambda \cdot f$, dat f even is, en weer de definitie van $\lambda \cdot f$. Dus $\lambda \cdot f$ zit in V^+. We hebben nu alle drie eisen nagegaan, dus V^+ is een deelruimte.

(2) We gaan de drie eisen voor deelruimtes na. Laat $V^- \subset \mathbb{R}^\mathbb{R}$ de deelverzameling van oneven functies zijn. De nulfunctie is oneven, dus $0 \in V^-$. Gesloten onder optelling. Laat f en g in V^-, dan geldt, voor alle $x \in \mathbb{R}$:

$$\begin{aligned} (f+g)(-x) &= f(-x) + g(-x) = -f(x) + (-g(x)) \\ &= -(f(x) + g(x)) = (f+g)(x), \end{aligned}$$

waarbij we achtereenvolgens hebben gebruikt: de definitie van $f + g$, dat f en g oneven zijn, een rekenregel voor reële getallen, en weer de definitie van $f + g$. Dus $f + g$ zit in V^-. Gesloten onder scalairvermenigvuldiging. Laat $\lambda \in \mathbb{R}$ en $f \in V^-$. Dan geldt, voor alle $x \in \mathbb{R}$:

$$(\lambda\cdot f)(-x) = \lambda\cdot(f(-x)) = \lambda\cdot(-f(x)) = -(\lambda\cdot f)(x),$$

waarbij we achtereenvolgens hebben gebruikt: de definitie van $\lambda\cdot f$, dat f oneven is, en weer de definitie van $\lambda\cdot f$. Dus $\lambda\cdot f$ zit in V^-. We hebben nu alle drie eisen nagegaan, dus V^- is een deelruimte.