Uitwerkingen werkcollege 10.

7.4.3 We have $\dim(U_1 \cap U_2) = 0$, so by Theorem 7.55 we have

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 \geq \dim V.$$

From the inclusion $U_1 + U_2 \subset V$ we also have $\dim(U_1 + U_2) \leq \dim V$, so we get $\dim(U_1 + U_2) = \dim V$ and Lemma 7.54 implies $U_1 + U_2 = V$. Together with $U_1 \cap U_2 = \{0\}$, this shows that U_1 and U_2 are complementary subspaces in V.

7.4.4 Hier zijn vele oplossingen mogelijk. Een strategie om er één te vinden is om Stelling 7.55 2 maal te gebruiken om een correcte formule af te leiden voor de dimensie van $U_1 + U_2 + U_3$, en dan te kijken waar die verschilt van de (niet correcte) formule in de opgave. Dus, 2 maal Stelling 7.55 toepassen geeft:

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim((U_1 + U_2) + U_3)\\ &= \dim(U_1 + U_2) + \dim(U_3) - \dim((U_1 + U_2) \cap U_3)\\ &= \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) + \dim(U_3) - \dim((U_1 + U_2) \cap U_3)\\ &= \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim((U_1 + U_2) \cap U_3).\end{aligned}$$

Hierin komt de term $\dim((U_1 + U_2) \cap U_3)$ voor, en dat kan tot het idee leiden om eens drie lijnen in een vlak te proberen: $F = \mathbb{R}$, $V = \mathbb{R}^2$, $U_1 = L(1, 0)$, $U_2 = L(0, 1)$ en $U_3 = L(1, 1)$. Dan geldt de gevraagde ongelijkheid inderdaad: links staat dan 2, en rechts 3.

7.4.5 We weten dan U_1+U_2 een deelruimte is van V en dat dus $\dim(U_1+U_2) \leq 10$. Stelling 7.56 geeft dan:

$$\begin{aligned}\dim(U_1 \cap U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2)\\ &= 6 + 7 - \dim(U_1 + U_2) \geq 13 - 10 = 3.\end{aligned}$$

8.1.2 (1) For each i, the map $S_i = \mathrm{ev}_{\alpha_i}$ that is evaluation at α_i is a linear map (cf. Example 4.31). The map S sends $f \in F[x]_n$ to $(S_1(f), \ldots, S_{n+1}(f))$, so S is linear by Exercise 4.4.3 (alternatively, one can of course check the requirements for being a linear map directly).

(2) Geïnspireerd door Voorbeeld 8.4 bekijken we voor elke i in $\{1, \ldots, n+1\}$ het element p_i van $F[x]_n$ gegeven door:

$$p_i = (x - \alpha_1) \cdots (x - \alpha_{n+1})/(x - \alpha_i) = \prod_{j \neq i} (x - \alpha_j).$$

Dan geldt

$$S(p_i) = \lambda_i \cdot e_i, \quad \text{met } \lambda_i = \prod_{j \neq i} (\alpha_i - \alpha_j) \neq 0.$$

Hieruit volgt dat de deelruimte $\mathrm{im}(S)$ alle e_i in F^{n+1} bevat en dus gelijk is aan F^{n+1}. Dus is S surjectief.

(3) Since we have $\dim F[x]_n = n + 1 = \dim F^{n+1}$, and S is surjective by part (2), Corollary 8.5 shows that S is an isomorphism.

(4) The conditions stated for f_i are equivalent to $S(f_i) = e_i$, where e_i denotes the i-th standard vector of F^{n+1}. Therefore, this follows from the fact that S is a bijection.

(5) The map S^{-1} sends the standard basis $(e_1, \dots, e_{n+1})$ to $(f_1, f_2, \dots, f_{n+1})$, so the latter is a basis for $F[x]_n$ by Proposition 7.31.

(6) We have already seen in part (2) that $S(p_i) = \lambda_i \cdot e_i$. Hence:

$$\begin{aligned} f_i &= \lambda_i^{-1} \cdot p_i \\ &= \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)}{(\alpha_i-\alpha_0)(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)} \\ &= \prod_{\substack{0 \le j \le n \\ j \ne i}} \left(\frac{x-\alpha_j}{\alpha_i-\alpha_j} \right). \end{aligned}$$

8.1.4 (1) Let $\tilde{g}$ denote the restriction of g to the subspace $\operatorname{im} f \subset V$. Then we have $\operatorname{im}(g \circ f) = \operatorname{im} \tilde{g}$ (do you see why?), so $\operatorname{rk}(g \circ f) = \operatorname{rk} \tilde{g}$. Since the domain of $\tilde{g}$ is $\operatorname{im} f$, Theorem 8.3 gives

$$\operatorname{rk} \tilde{g} = \dim(\operatorname{im} f) - \dim \ker \tilde{g} \le \dim \operatorname{im} f = \operatorname{rk} f,$$

with equality if and only if $\ker \tilde{g} = \{0\}$. We have $\ker \tilde{g} = \ker g \cap \operatorname{im} f$, so equality does indeed hold if g is injective.
To show the "But not only if" part, we give an example where g is not injective and still $\operatorname{rk}(g \circ f) = \operatorname{rk} f$. We want $\tilde{g}$ to be injective but g not. We can take $F = \mathbb{R}$, $U = \{0\}$, $V = \mathbb{R}$, $W = \{0\}$ and for f and g the only linear maps that exist, the zero-maps. For those who want a more interesting example: $U = \mathbb{R}$, $V = \mathbb{R}^2$, and $W = \mathbb{R}$, with, for all $x \in V$, $f(x) = (x, 0)$, and for all $(x, y) \in V$, $g(x, y) = x$.

(2) Note that the inclusion $f(U) = \operatorname{im} f \subset V$ yields $\operatorname{im}(g \circ f) = g(f(U)) \subset g(V) = \operatorname{im} g$, so by Lemma 7.54 we get $\operatorname{rk}(g \circ f) = \dim g(f(U)) \le \dim g(V) = \operatorname{rk} g$. If f is surjective, then we have $f(U) = V$, so all inclusions and inequalities are equalities.
To show the "But not only if" part, we give an example where f is not surjective and still $\operatorname{rk}(g \circ f) = \operatorname{rk} g$. We can take $F = \mathbb{R}$, $U = \{0\}$, $V = \mathbb{R}$, $W = \{0\}$ and for f and g the only linear maps that exist, the zero-maps. For those who want a more interesting example: $U = \mathbb{R}$, $V = \mathbb{R}^2$, and $W = \mathbb{R}$, with, for all $x \in V$, $f(x) = (x, 0)$, and for all $(x, y) \in V$, $g(x, y) = x$.

8.2.15.5.4 Rotation over α is an isomorphism: its inverse is rotation over $-\alpha$. So the image of ρ is $\mathbb{R}^2$, so the rank of ρ, and thus of the associated matrices, is 2.

5.5.5 The ranks are 2, 2, 3, 2, and 2, respectively (see Exercise 6.2.1).

8.2.4 Set $U = L(S)$. By Proposition 3.33, we have $S^\perp = L(S)^\perp = U^\perp$. Hence, this follows from Proposition 8.20.

8.2.6 (1) $\operatorname{rk}(AB) = \dim \operatorname{im}(AB)$, en $\operatorname{im}(AB)$ is een deelruimte van $\operatorname{im}(A)$. Lemma 7.54 geeft dat $\operatorname{rk}(AB) \le \operatorname{rk}(A)$. Als $\operatorname{rk}(B) = m$, dan is $B\colon F^n \to F^m$ surjectief, dus geldt $\operatorname{im}(AB) = A(B(F^n)) = A(F^m) = \operatorname{im}(A)$, dus rk$(AB)$ = rk(A). Om de 'not only if' te laten zien, willen we een voorbeeld waarin rk(AB) = rk(A) en rk$(B) < m$. We nemen $F = \mathbb{Q}$, $n = m = l = 1$, en $A = 0$ en $B = 0$.

(2) We hebben dat $\ker(B)$ een deelruimte is van $\ker(AB)$, dus $\dim(\ker(B)) \leq \dim(\ker(AB))$. De rangstelling (Stelling 8.3) geeft dan

$$\mathrm{rk}(B) = n - \dim(\ker(B)) \geq n - \dim(\ker(AB)) = \mathrm{rk}(AB).$$

Als rk$(A) = m$ dan is A injectief vanwege de rangstelling en dan geldt $\ker(AB) = \ker(B)$. Voor de 'not only if' nemen we hetzelfde voorbeeld als in (1).

(3) Dit volgt uit (1).

(4) Dit volgt uit (2).

8.3.1 Since $V^\perp$ is generated by x_3 and x_5, we find

$$U \cap V = U \cap (V^\perp)^\perp = \{u \in U \,:\, \langle u, x_3\rangle = \langle u, x_5\rangle = 0\}.$$

Every $u \in U$ can be written as $u = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. As in Example 8.28, the equations $\langle u, x_3\rangle = \langle u, x_5\rangle = 0$ are then equivalent to $(\lambda_1, \lambda_2, \lambda_3)$ lying in the kernel of the matrix

$$\begin{pmatrix} \langle u_1, x_3\rangle & \langle u_2, x_3\rangle & \langle u_2, x_3\rangle \\ \langle u_1, x_5\rangle & \langle u_2, x_5\rangle & \langle u_2, x_5\rangle \end{pmatrix} = \begin{pmatrix} 7 & 7 & -3 \\ 7 & 7 & -3 \end{pmatrix}.$$

Its kernel is generated by $(-1, 1, 0)$ and $(3, 0, 7)$, which correspond to the vectors $-u_1 + u_2$ and $3u_1 + 7u_3$, so these two vectors generate $U \cap V$.

8.3.2 The intersection is generated by $(1, 0, 0, 1) \in F^4$.