Uitwerkingen werkcollege 12.

9.3.1 (1)

$$[T]^{E_2}_{E_3} = \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$$

(2)

$$\begin{aligned}[T]^B_C &= [\mathrm{id}]^{E_3}_C \cdot [T]^{E_2}_{E_3} \cdot [T]^B_{E_2} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ -4 & 2 & -3 \\ -6 & 3 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 4 \\ -39 & -9 \\ -60 & -15 \end{pmatrix}\end{aligned}$$

9.4.1 (1) $1' = 0$ and $(1+x)' = 1$ and $(1+x+x^2)' = 1+2x = 2(1+x)-1$ and $(1+x+x^2+x^3)' = 1+2x+3x^2 = 3(1+x+x^2)-(x+1)-1$, so we get

$$[T]^B_B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

(2) We have

$$[T]^C_C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad [\mathrm{id}]^B_C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so

$$[T]^B_B = [\mathrm{id}]^C_B \cdot [T]^C_C \cdot [\mathrm{id}]^B_C = ([\mathrm{id}]^B_C)^{-1} \cdot [T]^C_C \cdot [\mathrm{id}]^B_C$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

9.4.2 (1) The line equals $L = L(a)$ with $a = (1, 2)$. The projection of e_i on L equals $\frac{\langle a, e_i \rangle}{\langle a, a \rangle} \cdot a$, so we have $\pi(e_1) = \frac{1}{5} a$ and $\pi(e_2) = \frac{2}{5} a$, so

$$[\pi]^B_B = \tfrac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} .$$

(2) Take $v_1 = a$ and $v_2 = (2, -1)$. Then

$$[\pi]^C_C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

(3)

$$\begin{aligned}[\pi]^B_B &= [\mathrm{id}]^C_B \cdot [\pi]^C_C \cdot [\mathrm{id}]^B_C = [\mathrm{id}]^C_B \cdot [\pi]^C_C \cdot ([\mathrm{id}]^C_B)^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \tfrac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\end{aligned}$$

9.4.3 (1) The vector $a = (1, 3, -2)$ is a normal of V. Since V contains 0, we have $V = a^\perp$. We have $\pi(e_i) = e_i - \frac{\langle a, e_i\rangle}{\langle a, a\rangle} \cdot a$, so $\pi(e_1) = e_1 - \frac{1}{14}a = \frac{1}{14}(13, -3, 2)$ and $\pi(e_2) = e_2 - \frac{3}{14}a = \frac{1}{14}(-3, 5, 6)$ and $\pi(e_3) = e_3 - \frac{-2}{14}a = \frac{1}{7}(1, 3, 5)$. Putting these vectors as columns in a matrix, we get

$$[\pi]_B^B = \frac{1}{14} \cdot \begin{pmatrix} 13 & -3 & 2 \\ -3 & 5 & 6 \\ 2 & 6 & 10 \end{pmatrix}$$

(2) We take $v_3 = a$. We have $V = a^\perp$, so V is equal to the kernel of the 1×3 matrix with a as its row. Generators are $v_1 = (-3, 1, 0)$ and $v_2 = (2, 0, 1)$. These do indeed form a basis for V, which has dimension 2. We get

$$[\pi]_C^C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$\begin{aligned} [\pi]_B^B &= [\mathrm{id}]_B^C \cdot [\pi]_C^C \cdot [\mathrm{id}]_C^B = [\mathrm{id}]_B^C \cdot [\pi]_C^C \cdot ([\mathrm{id}]_B^C)^{-1} \\ &= \begin{pmatrix} -3 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -3 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}^{-1} \\ &= \frac{1}{14} \cdot \begin{pmatrix} 13 & -3 & 2 \\ -3 & 5 & 6 \\ 2 & 6 & 10 \end{pmatrix}. \end{aligned}$$

9.5.1 $\mathrm{Tr}(M_1) = -10$ and $\mathrm{Tr}(M_2) = -10$ and $\mathrm{Tr}(M_3) = 3$.

9.5.2 Reflexive: For $Q = I_n$, we have $A = QAQ^{-1}$, so A is similar to itself.

Symmetry: If A is similar to B, then there is an invertible $Q \in \mathrm{Mat}(n, F)$ such that $A = QBQ^{-1}$. Then for $P = Q^{-1}$ we have $B = PAP^{-1}$, so B is similar to A.

Transitive: Suppose A is similar to B, and B is similar to C. Then there are invertible matrices P and Q such that $A = PBP^{-1}$ and $B = QCQ^{-1}$. Then we have $A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1}$, so A is also similar to C.

This proves that similarity defines an equivalence relation.

10.1.1 De determinanten zijn -4, 0, 16, 17, -10.

10.1.2 Write the upper triangular matrix as $A = (a_{ij})_{i,j=1}^n$. We use induction on n. For $n = 1$ (or even $n = 0$), the statement is true, so suppose $n > 1$. Because A is upper triangular, we have $a_{nj} = 0$ for $j < n$, so the expansion of the determinant along the n-th row yields

$$\det A = a_{nn} \cdot \det A_{nn}, \tag{1}$$

where A_{nn} is the matrix obtained from A by leaving out the n-th row and the n-th column. The matrix A_{nn} is upper triangular, with its diagonal entries being exactly the same as those of A, except for the last entry a_{nn}. By the induction hypothesis, the determinant $\det A_{nn}$ is the product of these diagonal entries of A_{nn}. From (1) we then find that $\det A$ is the product of all diagonal entries of A.

For lower triangular matrices we can do the same, except that we use expansion along the first row. Alternatively, we wait until the next section and use the identity $\det A = \det A^\top$, and the fact that the transpose of a lower triangular matrix is an upper triangular matrix.

10.1.3 De determinant is $xy(x-1)(y-1)(y-x)$. Je kunt die krijgen door de eerste kolom x keer van de de 2e af te trekken, en y keer van de 3e; dan kun je factoren x, y, $x-1$ en $y-1$ naar voren halen, en de 2 bij 2 matrix die dan over is geeft de factor $y-x$.