Uitwerkingen werkcollege 14.

11.2.1 The constant term of the characteristic polynomial P_A is the value

$$P_A(0) = \det(0 \cdot I - A) = \det(-A) = (-1)^n \det(A).$$

For the coefficient of t^{n-1}, we use induction. For $n = 1$ this is trivially equal to $-\operatorname{Tr} A$, so assume $n > 1$. We compute the determinant of $C = tI_n - A$ by expansion along the first row. We write $C = (c_{ij})_{i,j}$ and get

$$P_A(t) = \det(tI_n - A) = \det C = \sum_{j=1}^{n} (-1)^{1+j} c_{1j} \cdot \det C_{1j},$$

where C_{1j} denotes the matrix obtained from C by deleting the first row and the j-th column. For $j > 1$, the matrix C_{1j} contains only $n-2$ entries that are linear in t, while the rest of the entries are constant, so $\det C_{1j}$ has degree at most $n-2$. This implies that the coefficient of t^{n-1} in P_A is equal to the coefficient of t^{n-1} in the term for $j = 1$, which by the induction hypothesis is equal to

$$\begin{aligned} c_{11} \cdot \det C_{11} &= (t - a_{11}) \cdot \det(tI_{n-1} - A_{11}) = (t - a_{11}) \cdot P_{A_{11}}(t) \\ &= (t - a_{11})\big(t^{n-1} - \operatorname{Tr}(A_{11})t^{n-2} + \dots\big) = t^n - (\operatorname{Tr} A_{11} + a_{11})t^{n-1} + \dots \end{aligned}$$

So the coefficient of t^{n-1} is $-a_{11} - \operatorname{Tr} A_{11} = -\operatorname{Tr} A$.

11.2.2 Here it is a good idea to choose a basis of $\mathbb{R}^3$ that is adapted to the question. So take $v_1 \in \mathbb{R}^3$ such that $V = v_1^\perp$, and let v_2 and v_3 form a basis of V, then, as V and $L(v_1)$ are complementary, v_1, v_2 and v_3 form a basis of $\mathbb{R}^3$. The matrix of s with respect to this basis is diagonal, with diagonal entries $(-1, 1, 1)$. Therefore, $P_s(t) = (t-1)^2(t+1)$.

11.2.3 (1) Characteristic polynomial is $t^2 - 4$.
Eigenvalues are 2 and -2.
Basis of eigenspace for $\lambda = -2$ is $\big((1, -1)\big)$.
Basis of eigenspace for $\lambda = 2$ is $\big((1, -2)\big)$.

(2) Characteristic polynomial is $(t-3)^2$.
Eigenvalue is 3.
Basis of eigenspace for $\lambda = 3$ is $\big((1, 2)\big)$.

(3) Characteristic polynomial is $(t+1)(t-3)^2$.
Eigenvalues are -1 and 3.
Basis of eigenspace for $\lambda = -1$ is $\big((1, 0, -1)\big)$.
Basis of eigenspace for $\lambda = 3$ is $\big((2, 0, -1), (0, 1, 0)\big)$.

(4) Characteristic polynomial is $(t-2)^3$.
Eigenvalue is only 2.
Basis of eigenspace for $\lambda = 2$ is $\big((1, -2, 0), (0, 0, 1)\big)$.

(5) Characteristic polynomial is $(t-1)^2(t-2)(t+3)$.
Eigenvalues are $1, 2$, and -3.
Basis of eigenspace for $\lambda = 2$ is $\big((1, -1, 0, 0)\big)$.
Basis of eigenspace for $\lambda = -3$ is $\big((0, 0, 0, 1)\big)$.
Basis of eigenspace for $\lambda = 1$ is $\big((0, 0, 1, 0)\big)$.

(6) Characteristic polynomial is $(t-1)^2(t-2)^2$.
Eigenvalues are 1 and 2.
Basis of eigenspace for $\lambda = 2$ is $\big((1,0,-2,0)\big)$.
Basis of eigenspace for $\lambda = 1$ is $\big((0,0,1,0)\big)$.

11.3.1 For $k > 0$ this was already proved in the text. For $k = 0$ both sides are equal to I_n, so we may assume $k < 0$. Then by Proposition 5.25, the inverse of PDP^{-1} is

$$\left(PDP^{-1}\right)^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}.$$

Hence, we find

$$\left(PDP^{-1}\right)^{k} = \left(\left(PDP^{-1}\right)^{-1}\right)^{-k} = \underbrace{(PD^{-1}P^{-1})(PD^{-1}P^{-1})\cdots(PD^{-1}P^{-1})}_{-k}$$
$$= P(D^{-1})^{-k}P^{-1} = PD^kP^{-1}.$$

11.3.2 (1) The first is not diagonalisable, because the only eigenvalue is 1 and the corresponding eigenspace is 1-dimensional, namely generated by $e_1 = (1,0)$.

(2)
$$P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

(3)
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ -2 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

11.3.3 (1) Yes, by Proposition 11.25.
$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

(2) Yes, by Proposition 11.25.
$$P = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(3) Yes, by Proposition 11.25.
$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

(4) No: only eigenvalue is 3, with algebraic multiplicity 2 and geometric multiplicity 1.

(5) Yes.
$$P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(6) No: only eigenvalue is 2, with algebraic multiplicity 3 and geometric multiplicity 2.

(7) No: the geometric multiplicity of $\lambda = 1$ is 1 and the algebraic multiplicity is 2.
(8) No: the geometric multiplicity of $\lambda = 1$ is 1 and the algebraic multiplicity is 2.

11.3.11 Let ρ denote the rotation over θ. If $\rho(v) = \lambda v$ for some nonzero $v \in \mathbb{R}^2$ and some $\lambda \in \mathbb{R}$, then ρ sends the line $L = L(v)$ to itself, so the angle is either 0 or 180° (in the first case all eigenvalues are 1, in the second case they are -1).

11.3.12 (1) Since all columns are the same, the column space $C(N_n)$ is generated by the vector $a = (1, 1, 1, \ldots, 1)$, so $\operatorname{rk} N_n = \dim C(N_n) = 1$, and hence $\dim \ker N_n = n - \operatorname{rk} N_n = n - 1$.
(2) If v is an eigenvector with eigenvalue λ, then $\lambda v = N_n v \in \operatorname{im} N_n = L(a)$, so λv is a multiple of a. Then either the eigenvalue λ is 0, or we get $v \in L(a)$, say $v = \mu a$ for some $\mu \in \mathbb{R}$ and then $N_n v = \mu N_n a = \mu n a = nv$, so the eigenvalue λ is n.
(3) The corresponding eigenspaces are $E_n(N_n) = L(a)$ and $E_0(N_n) = \ker N_n$, which has dimension $n - 1$. Hence, the dimensions of the eigenspaces add up to n, so N_n is diagonalisable.
(4) Since N_n is diagonalisable, the algebraic and geometric multiplicities of the eigenvalues 0 and n agree, so they are $n-1$ and 1, repectively, so the characteristic polynomial is $P_{N_n}(t) = (t-n)(t-0)^{n-1} = t^n - nt^{n-1}$.
(5) Substituting $\lambda = 1$ into P_{N_n}, we find

$$\begin{aligned}\det M_n &= \det(N_n - I_n) = \det(-I_n(I_n - N_n)) = \det(-I_n) \cdot \det(1 \cdot I_n - N_n) \\ &= (-1)^n P_{N_n}(1) = (-1)^n (1 - n).\end{aligned}$$