Uitwerkingen werkcollege 3.

2.1.2 Firstly note that for the zero vector $0_V = (0,0,0) \in \mathbb{R}^3$ we have that $0_V \in V$, as $0+0+0=0$.

We need to check that if we "add" two elements $x, y \in V$ with the usual operation $\oplus$, we actually get an element of V. This is indeed the case because if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ satisfy $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$, then the sum $z = x \oplus y$ equals (z_1, z_2, z_3) with $z_i = x_i + y_i$ for $1 \leq i \leq 3$, and we have

$$\begin{aligned} z_1 + z_2 + z_3 &= (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) \\ &= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0 = 0, \end{aligned}$$

so also $z \in V$.

We also need to check that every scalar multiple of any element $x = (x_1, x_2, x_3) \in V$ is again contained in V. This is indeed the case, because for every $\lambda \in \mathbb{R}$ we have $\lambda \odot x = (\lambda x_1, \lambda x_2, \lambda x_3)$ and if $x_1 + x_2 + x_3 = 0$, then $\lambda x_1 + \lambda x_2 + \lambda x_3 = \lambda(x_1 + x_2 + x_3) = 0$, so $\lambda \odot x \in V$.

Now that we know that the operations $\oplus$ and $\odot$ are indeed well defined, we can check whether together with the zero $0_V = (0,0,0)$, they satisfy the eight axioms of Definition 2.1.

To check that for all $x, y \in V$ we have $x \oplus y = y \oplus x$, we note that we already knew this for all $x, y \in \mathbb{R}^3$, so it certainly holds for all $x, y \in V$, because V is a subset of $\mathbb{R}^3$. Similarly, all axioms, except for the fourth, hold for all elements in $\mathbb{R}^3$, so certainly for all elements in V.

It remains to check the fourth axiom. For $x = (x_1, x_2, x_3) \in V$ we define $x' = (-x_1, -x_2, -x_3)$, of which the sum of coordinates is $(-x_1) + (-x_2) + (-x_3) = -(x_1 + x_2 + x_3) = 0$, so we have $x' \in V$. Of course, we have $x \oplus x' = (x_1 + (-x_1), x_2 + (-x_2), x_3 + (-x_3)) = (0,0,0) = 0_V$, so also the fourth axiom is satisfied. It follows that V, together with the usual coordinate-wise addition and scalar multiplication and 0_V as zero vector is indeed a vector space.

2.1.3 In this case, V is not a vector space, because it is not closed under addition. To show this, it suffices to give two elements $x, y \in V$ with $x \oplus y \notin V$. For example, the elements $x = (1,0,0)$ and $y = (0,1,0)$ are contained in V, but $x \oplus y = (1,1,0)$ is not. [Alternatively, one can show that V is not closed under scalar multiplication: if $x \in V$, then $2x \notin V$. Or one notes that the suggested zero vector is not contained in V.]

2.1.4

a	b	$a \odot b$	$a \oplus b$
F	F	$\times$	$\times$
F	V	V	$\times$
V	V	$\times$	V

2.2.9 (3) Yes, all axioms can be checked.

(4) Yes, all axioms can be checked.

(5) No, the sum of two elements is not even in V, and the scalar multiples (by scalars other than 1) are not in V either, so we have no addition and scalar multiplication on V. So we can not even phrase any of the axioms, except for axiom (6).

(6) Yes, all axioms can be checked.

(7) No, the sum of two elements is not even in V, and the scalar multiples (by scalars other than 1) are not in V either, so we have no addition and scalar multiplication on V. So we can not even phrase any of the axioms, except for axiom (6).

2.2.10 By Lemma 1.25, the set $a^\perp$ is closed under addition and scalar multiplication: for every $x, y \in a^\perp$ and $\lambda \in \mathbb{R}$, we have $x \oplus y \in a^\perp$ and $\lambda \odot x \in a^\perp$. Therefore, we do indeed have a well defined addition and scalar multiplication on $a^\perp$. As in Exercise 2.1.2, we note that all axioms except axiom (4) state that some property has to hold for all vectors (or pairs or triples of vectors) in $a^\perp$ and perhaps all scalars (or pairs of scalars). Given that we already know these properties hold for all (pairs or triples of) vectors in $\mathbb{R}^n$, they certainly hold for all (pairs or triples of) vectors in $a^\perp$. As for axiom (4), for every $x \in a^\perp$ we know that $x' = (-1) \cdot x$ is contained in $a^\perp$, and $x + x' = (1 + (-1))x = 0 \cdot x = 0$, so also axiom (4) is satisfied. This finishes the proof. Cf. Lemma 3.2.

2.2.12 For f and g in $V^X = \mathrm{Map}(X, V)$ we define $f + g$ in V^X by defining, for all x in X, $(f + g)(x) = f(x) + g(x)$ (pointwise addition). Similarly, we define, for f in V^X and for λ in F, and for all x in X, $(\lambda\cdot f)(x) = \lambda\cdot(f(x))$ (pointwise multiplication). One can then check that all 8 axioms for vector spaces hold in this case.

2.2.14 Laat $a = (a_n)_{n\geq 0}$ en $b = (b_n)_{n\geq 0}$ elementen van S zijn. De termsgewijze som $a+b$ is dan de rij gegeven door $(a+b)_n = a_n + b_n$. Voor deze rij geldt, voor alle $n \geq 0$:

$$\begin{aligned}(a + b)_{n+2} &= a_{n+2} + b_{n+2} = (a_{n+1} + a_n) + (b_{n+1} + b_n) = \\ &= (a_{n+1} + b_{n+1}) + (a_n + b_n) = (a + b)_{n+1} + (a + b)_n,\end{aligned}$$

dus is $a + b$ ook in S. Laat nu $a = (a_n)_{n\geq 0}$ in S en $\lambda \in \mathbb{R}$. Dan is het termsgewijze product $\lambda\cdot a$ gegeven door $(\lambda\cdot a)_n = \lambda\cdot a_n$. Voor deze rij geldt, voor alle $n \geq 0$:

$$\begin{aligned}(\lambda\cdot a)_{n+2} &= \lambda\cdot a_{n+2} = \lambda\cdot(a_{n+1} + a_n) = \\ &= \lambda\cdot a_{n+1} + \lambda\cdot a_n = (\lambda\cdot a)_{n+1} + (\lambda\cdot a)_n,\end{aligned}$$

dus is $\lambda\cdot a$ ook in S. We moeten nu laten zien dat S, met deze optelling en scalairvermenigvuldiging, een $\mathbb{R}$-vectorruimte is. Aan alle 8 axioma's behalve de 4-de is voldaan omdat de gevraagde gelijkheden termsgewijs gelden (we kunnen ook zeggen dat S een deelverzameling is van $\mathrm{Map}(\mathbb{N}, \mathbb{R})$, en dat de gevraagde gelijkheden gelden in deze grotere vectorruimte). Voor axioma 4: gegeven $a = (a_n)_{n\geq 0}$, laat $a' = (-1)\cdot a$. Dan is a' in S (zie hierboven, met $\lambda = -1$), en geldt $a + a' = 0$.

Opmerking: met de kennis van Hoofdstuk 3 kunnen we zeggen dat S een deelruimte is van $\mathrm{Map}(\mathbb{N}, \mathbb{R})$ en dus een vectorruimte is.

2.3.2 For (3), we have

$$\lambda \odot 0_V = \lambda \odot (0_V \oplus 0_V) = (\lambda \odot 0_V) \oplus (\lambda \odot 0_V),$$

where the first equality follows from axiom (3) and the second from axiom (7). Proposition 2.17 applied to the case $0' = z = \lambda \odot 0_V$ then yields $\lambda \odot 0_V = 0_V$.

Statement (4) follows from statements (2) and (3) with $x = 0_V$ and $\lambda = -1$.

For statement (6), let $\lambda \in F$ and $x \in V$, then we have

$$\begin{aligned} -(\lambda \odot x) &= (-1) \odot (\lambda \odot x) && \text{by prop 2.20(2)} \\ &= (-1 \cdot \lambda) \odot x && \text{by distributivity, axiom 5} \\ &= (-\lambda) \odot x \end{aligned}$$

and

$$\begin{aligned} (-\lambda) \odot x &= (-1 \cdot \lambda) \odot x \\ &= (\lambda \cdot -1) \odot x && \text{by commutativity in field } F \\ &= \lambda \odot (-1 \odot x) && \text{by distributivity, axiom 5} \\ &= \lambda \odot (-x) && \text{by prop 2.20(2).} \end{aligned}$$

For statement (7), let $x, y, z \in V$. Suppose $z = x \ominus y$, then

$$z \oplus y = (x \ominus y) \oplus y = (x \oplus (-y)) \oplus y = x \oplus ((-y) \oplus y) = x \oplus 0_V = x.$$

Suppose $x = y \oplus z$, then

$$x \ominus y = (y \oplus z) \ominus y = (z \oplus y) \ominus y = (z \oplus y) \oplus (-y) = z \oplus (y \oplus (-y)) = z \oplus 0_V = z.$$

2.3.3 False. For the vector $x' = (-1) \odot x$ we do indeed have

$$x + x' = (1 + (-1)) \odot x = 0 \odot x,$$

where the 0 obviously denotes the scalar zero in F. However, we used axiom (4) to prove $0 \odot x = 0_V$, so without axiom (4) we can not conclude $x \oplus x' = 0_V$.