Uitwerkingen werkcollege 9.

7.1.1 (1) Yes. (2) No.

7.1.4 Let A be the 5×6 matrix with $v_1, \ldots, v_6$ as columns. Then there are more columns than rows, so not every column of a row echelon form for A contains a pivot, so the kernel of f_A is nontrivial, so the elements $v_1, \ldots, v_6$ are linearly dependent by Proposition 7.10 and/or Corollary 7.11. See also Exercise 6.3.5.

7.1.8 Suppose $0 \leq j \leq n$. For all $0 \leq i < j$ we have $f_i(a_j) = 0$, so for every linear combination f of $f_0, f_1, \ldots, f_{j-1}$ we have $f(a_j) = 0$. The function f_j satisfies $f_j(a_j) = 1$, so it is not a linear combination of $f_0, f_1, \ldots, f_{j-1}$. This holds for all j, so by part (1) of Proposition 7.15, the sequence $(f_0, f_1, \ldots, f_n)$ is indeed linearly independent.

7.2.1 Many answers are possible in this case. While one could use Proposition 7.22, we will use Proposition 7.26 (or Lemma 7.24). We don't list the matrices and their row echelon forms here, but only the conclusion.
(1) (v_1, v_2), which generates $\mathbb{R}^2$.
(2) (v_1, v_2), note that $5v_3 = v_1 + 2v_2$, so $v_3 \in L(v_1, v_2)$.
(3) (v_1, v_2), note that $4v_3 = v_1 + v_2$, so $v_3 \in L(v_1, v_2)$.
(4) (v_1, v_2), note that $v_3 = -v_1 + 2v_2 \in L(v_1, v_2)$.
(5) (v_1, v_2, v_3).

7.2.3 We select the columns of A corresponding to the columns of a row echelon form of A that contain a pivot (see Proposition 7.26). In Exercise 6.3.3 we have seen row echelon forms for each of the four matrices, so we can choose:
(1) for the first matrix: the first two columns,
(2) for the second (the 3×3 matrix): the first two columns,
(3) for the third (the 3×5 matrix): the first, second, and fourth column,
(4) for the last matrix: the first, second, and fourth column.

7.2.4 None of the first three polynomials is a linear combination of the previous, because their degrees are increasing. The polynomial f_4 is also not a linear combination of the previous. (Why? Check this with a computation!) The polynomials f_5 and f_6 are linear combinations of the previous, as we have $f_5 = f_1 + f_2 - f_3 + f_4$ and $f_6 = f_1 + f_3 - f_4$. Therefore, by Proposition 7.24, the polynomials f_1, f_2, f_3, f_4 form a basis for U.

Andere oplossing. Merk op dat alle f_i in $\mathbb{R}[x]_4$ zitten. Gebruik de basis $(1, x, x^2, x^3, x^4)$ van $\mathbb{R}[x]_4$ om de f_i als elementen van $\mathbb{R}^4$ te schrijven. Gebruik nu Remark 7.27: zet de f_i als kolommen in een matrix A, kijk waar de spillen staan in de ge-rij-trap-vormde matrix A', en neem die f_i. Dat geeft dat f_1, f_2, f_3, f_4 een basis van U vormen.

7.3.1 As in Example 7.31, we look at the matrix whose columns are the coefficients of the polynomials f_1, f_2, f_3 and the generators $1, x, x^2, x^3, x^4$, which is

$$A = \begin{pmatrix} 2 & -3 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A row echelon form of this matrix is

$$A' = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{1}{7} & 0 & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The first three columns of A' contain a pivot, so they are indeed linearly independent. The other two pivots, in the fifth and eighth column, correspond to x and x^4, so we can extend to a basis f_1, f_2, f_3, x, x^4.

7.3.3 (1) The hyperplane V is equal to the kernel of the 1×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix},$$

which is in row echelon form. By Proposition 7.22, a basis for this kernel is given by the elements obtained from Proposition 6.17, which are

$$w_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad w_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the dimension of V is 3, corresponding to the number of columns of A without a pivot.

(2)&(3) We have $\langle a, v_1 \rangle = \langle a, v_2 \rangle = 0$, so $v_1, v_2 \in a^\perp = V$. The matrix that has v_1, v_2, w_2, w_3, w_4 as columns is

$$B = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -3 & 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the last three columns generate V by part (1). A row echelon form for B is

$$B' = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first two columns contain a pivot, so v_1 and v_2 are indeed linearly independent (which could also be seen very quickly, as it is just two vectors and neither is a scalar multiple of the other). The only other column with a pivot is the fourth, which corresponds with w_3, so we can extend (v_1, v_2) to a basis (v_1, v_2, w_3) of V.

7.3.5 If all sequences of linearly independent elements have length bounded by m, then V is finitely generated by Proposition 7.53, so it has a finite basis and dimension, say $n = \dim V$. The n elements of a basis are linearly independent, so we find $n \leq m$ by the assumption.