# Training session 2 ‘Groups and symmetries in geometry’

**Give arguments for all your claims. Write complete sentences, including quantifiers.**

1. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a ring morphism (this means that $\sigma(0) = 0$ and $\sigma(1) = 1$ and, for all $x$ and $y$ in $\mathbb{R}$, we have $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$).
   (a) Show that for all $n \in \mathbb{Z}$, $\sigma(n) = n$.
   (b) Show that for all $q \in \mathbb{Q}$, $\sigma(q) = q$.
   (c) Show that $\forall x, y \in \mathbb{R}\,(x \leq y \implies \sigma(x) \leq \sigma(y))$. Hint: $x \leq y$ is equivalent to $y - x \geq 0$, and then consider which real numbers are squares.
   (d) Show that $\sigma = \mathrm{id}_{\mathbb{R}}$ (meaning that $\forall x \in \mathbb{R}\, \sigma(x) = x$).
   (e) Of the following list of properties of $\mathbb{R}$, which one(s) do you *not* need in the previous part?
      (a) ($\mathbb{R}$ is real closed) Every $x \in \mathbb{R}$ with $x \geq 0$ is a square.
      (b) ($\mathbb{R}$ is archimedean) For every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $n \geq x$.
      (c) ($\mathbb{R}$ is complete) Every Cauchy sequence in $\mathbb{R}$ converges.
      (d) (supremum property) Every non-empty bounded subset of $\mathbb{R}$ has a supremum.

2. Let $G := \mathrm{SU}(2)$, the group of $g \in \mathrm{M}_2(\mathbb{C})$ such that $\det(g) = 1$ and $\overline{g}^t \cdot g = 1_2$.
   (a) Show that an element $g \in \mathrm{M}_2(\mathbb{C})$ is in $\mathrm{SU}(2)$ if and only if (i): its columns form an orthonormal basis of $\mathbb{C}^2$ for the standard complex inner product, and (ii): $\det(g) = 1$.
   (b) Show that $\mathrm{SU}(2) = \{\left(\begin{smallmatrix} a & -\overline{b} \\ b & \overline{a} \end{smallmatrix}\right) : |a|^2 + |b|^2 = 1\}$.
   (c) Conclude that $\mathrm{SU}(2)$ is the unit 3-sphere in $\mathbb{C}^2 = \mathbb{R}^4$, and so is a 3-dimensional (real) manifold.
   (d) Compute the tangent space $T_{\mathrm{SU}(2)}(1_2)$, and show that the matrices $I$, $J$ and $K$ on page 5 of the lecture notes form a basis of it.
   (e) Verify by hand that $T_{\mathrm{SU}(2)}(1_2)$ is a sub-Lie-algebra of the real Lie-algebra $\mathrm{M}_2(\mathbb{C})$. This means: compute the commutators of $I$, $J$ and $K$ and verify that these are $\mathbb{R}$-linear combinations of $I$, $J$ and $K$.