Training session 3 'Groups and symmetries in geometry'

Give arguments for all your claims. Write complete sentences, including quantifiers.

1. Let n be in $\mathbb{N}$.
   (a) Prove that for all $A \in \mathrm{M}_n(\mathbb{C})$ the series $e^A := \sum_{m\geq 0} \frac{1}{m!}A^m$ converges in $\mathrm{M}_n(\mathbb{C})$. Hint: choose a norm on $\mathrm{M}_n(\mathbb{C})$, for example $\|A\| := \max\{|A_{i,j}| : 1 \leq i,j \leq n\}$, or any other norm (they are all equivalent).
   (b) Show that for A and B in $\mathrm{M}_n(\mathbb{C})$ with $AB = BA$ we have $e^{A+B} = e^A e^B$.
   (c) Give A and B in $\mathrm{M}_n(\mathbb{C})$ such that $e^{A+B} \neq e^A e^B$.
   (d) Show that $\det(e^A) = e^{\mathrm{tr}(A)}$. Hint: show that it suffices to prove it for diagonalisable A, and then reduce to diagonal A.

2. Let G be a Lie group. Show that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto (D\psi_g)(1)$ is a representation of G on $\mathfrak{g}$. Here $\psi_g\colon G \to G$, $h \mapsto ghg^{-1}$, and $(D\psi_g)(1)\colon \mathfrak{g} \to \mathfrak{g}$ is the derivative of ψ_g at the identity element 1 of G. See also Benthem's BSc thesis. First consider the case $G = \mathrm{GL}_n$.

   This question can be generalised as follows. Let X be a manifold and let G act on X, such that the map $G \times X \to X$ is a morphism of manifolds. Suppose that G fixes a point x in X, Then G acts on the tangent space $T_X(x)$ and this is a representation of G on $T_X(x)$.

3. Let G be a group, let X be a set, and $G \times X \to X$, $(g,x) \mapsto g{\cdot}x$ an action of G on X.
   (a) Let $\mathbb{C}^X$ be the set of all functions $f\colon X \to \mathbb{C}$. Show that $G \times \mathbb{C}^X \to \mathbb{C}^X$, $(g,f) \mapsto g\bullet f$, with, for all $x \in X$, $(g\bullet f)(x) = f(g^{-1}x)$, is an action of G on $\mathbb{C}^X$.
   (b) Show that the action in the previous part is linear, where addition and scalar multiplication in $\mathbb{C}^X$ are point-wise ($(f_1+f_2)(x) = f_1(x)+f_2(x)$, $(\lambda{\cdot}f)(x) = \lambda{\cdot}(f(x))$), hence makes $\mathbb{C}^X$ into a representation of G.
   (c) The $\mathbb{C}$-vector space $\mathbb{C}^X$ is even a $\mathbb{C}$-algebra: $(f_1{\cdot}f_2)(x) = f_1(x){\cdot}f_2(x)$ (point-wise multiplication). Show that the action in the previous part is by $\mathbb{C}$-algebra automorphisms: for each g in G, the map $\mathbb{C}^X \to \mathbb{C}^X$, $f \mapsto g\bullet f$ is an isomorphism of $\mathbb{C}$-algebras.

(d) Now we take $G = \mathrm{SU}(2)$ and $X = \mathbb{C}^2$, and the action is $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \left(\begin{smallmatrix} au-\bar{b}v \\ bu+\bar{a}v \end{smallmatrix}\right)$. In $\mathbb{C}^X = \mathbb{C}^{(\mathbb{C}^2)}$ we have the sub-$\mathbb{C}$-algebra $\mathbb{C}[x,y]$ of polynomial functions $\mathbb{C}^2 \to \mathbb{C}$, with $x(u,v) = u$ and $y(u,v) = v$. Compute, for $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ in $\mathrm{SU}(2)$, $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)\bullet x$ and $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)\bullet y$. You should find $\bar{a}x + \bar{b}y$ and $-bx + ay$.

(e) Conclude that the $\mathrm{SU}(2)$-action on $\mathbb{C}^X$ preserves $\mathbb{C}[x,y]$, and that

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \bullet (x^j y^k) = (\bar{a}x + \bar{b}y)^j \cdot (-bx + ay)^k \,.$$

Show that this agrees with the identities on page 10 of Benthem's BSc thesis.

4. On page 12 of Benthem's BSc thesis it is written that $\ker(\mathrm{SU}(2) \to \mathrm{SO}(3))$ is the intersection of $\mathrm{SU}(2) = \{x \in \mathbb{H} : N(x) = 1\}$ with the center $Z(\mathbb{H}) = \{x \in \mathbb{H} : \forall y \in \mathbb{H}\,, xy = yx\}$, and that that intersection is $\{1, -1\}$. We provide details.

(a) Prove that $Z(\mathbb{H}) = \mathbb{R}\cdot 1 \subset \mathbb{H}$.

(b) Prove that if $x \in \mathrm{SU}(2)$ is such that, for all $y \in \mathbb{R}\cdot I + \mathbb{R}\cdot J + \mathbb{R}\cdot K$, $xyx^{-1} = y$, then $x \in Z(\mathbb{H})$.

5. Prove the formula of Clebsch-Gordan, which means for us, prove that for d_1 and d_2 in $\mathbb{Z}_{\geq 0}$, there is an isomorphism of representations of $\mathrm{SU}(2)$:

$$\mathbb{C}[x,y]_{d_1} \otimes \mathbb{C}[x,y]_{d_2} \longrightarrow \bigoplus_{\substack{d=|d_1-d_2| \\ d \equiv d_1+d_2 \bmod 2}}^{d_1+d_2} \mathbb{C}[x,y]_d \,.$$

Hint: prove that both sides have the same character. Try first with small values for d_1 and d_2. Physicists use this to understand the total angular momentum (around some given direction) of an atom with 2 electrons; see wikipedia.

6. (a) Let g be in $\mathrm{O}(3)$, with $g \neq \mathrm{id}$. Show that the complex eigenvalues λ of g satisfy $|\lambda| = 1$, and that if λ is an eigenvalue, then so is $\bar{\lambda}$. Show that 1 or -1 is an eigenvalue of λ.

(b) Let g be in $\mathrm{SO}(3)$, with $g \neq \mathrm{id}$. Show that there is a unique $\phi \in [0, \pi]$ such that the complex eigenvalues of g are 1, $e^{i\phi}$ and $e^{-i\phi}$. Deduce from this that there is an oriented orthonormal basis v_1, v_2, v_3 of $\mathbb{R}^3$ such that g is the rotation about the line $\mathbb{R}\cdot v_3$ over the angle ϕ, and that with respect to the oriented basis $v_2, v_1, -v_3$ g is the rotation about $\mathbb{R}\cdot v_3$ over the angle $-\phi$.

(c) Make a character table for $\mathrm{SO}(3)$.