Examples of Weil-Deligne representations and of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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Theo discussed some abstract things, he described:

$$\lim_{\to K} H^1(\mathrm{Sh}_K(\mathrm{GL}_2(\mathbb{Q}), \mathbb{H}^{\pm})_{\overline{\mathbb{Q}}}, \mathcal{F}_{k,l}) = \bigoplus_{f \text{ newform of weight k}} \pi_f \otimes \rho_{f,l}^{\vee},$$

which has an action by $\mathrm{GL}_2(_f) \times G_{\mathbb{Q}}$ with $\mathcal{F}_{k,l} = \mathrm{Sym}^{k-2}(R^1\pi_*\overline{\mathbb{Q}_l})$. Theo proved that this exists. Here π_f is an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f)$ and $\rho_{f,l}$ is an irreducible $G_{\mathbb{Q}}$ representation of dimension 2. (This is also a way to state multiplicity one.)

For all primes p one has $\rho_{f,l}$ restricted to $G_{\mathbb{Q}_p}$, called $\rho^l_{f,p}$, and $\pi_{f,p}$, the latter being a representation of $GL_2(\mathbb{Q}_p)$.

Local Langlands says how $\pi_{f,p}$ and $\rho^l_{f,p}$ are related.

You want to know the theorem???? Here it is.

Theorem 1 (Langlands, Eichler-Shimura, Deligne, Carayol) *The representations $\pi_{f,p}$ and $\rho^l_{f,p}$, up to F-semisimplification, determine each other, under the assumption $l \neq p$.*

This is much stronger than multiplicity one. It is in particular an explicit rule. One usually wants to understand the Galois representations, so one can translate them to the automorphic side. In the unramified case, one asks what the eigenvalues of Frobenius are (can be read off from eigenvalues of Hecke operators). In the ramified places it is not so easy. That's where the theorem is useful. It is not true that the theorem only says that the L-functions are the same.

*Notes typed by Gabor Wiese.

At the interesting places in fact the p-factor of the L-function is just 1. So the statement about L-functions is valid all places, but gives little information at the bad places.

The proof uses a characterisation of $\pi_{f,p}$ and $\rho_{f,p}$ by ϵ-factors of a lot of twists. These ϵ-factors come from the constant in the functional equation. They come mostly from the ramification.

Why do we want to talk about Weil-Deligne represenations? Above we had to chose a prime l and work over $\overline{\mathbb{Q}_l}$; this can be avoided.

The right way to formulate local Langlands is to use Weil-Deligne representations and NOT Galois representations!!!

Situation: p prime as before, $\mathbb{Q}_p$, $\overline{\mathbb{Z}_p}$ integers in $\overline{\mathbb{Q}_p}$, $G_{\mathbb{Q}_p} \to G_{\overline{\mathbb{F}_p}} \cong \widehat{\mathbb{Z}}$ (via sending arithmetic Frobenius to 1). Take inverse image of $\mathbb{Z}$ and call it W_p, the *Weil group*. Take the map $d : W_p \to \mathbb{Z}$, which we call *degree*.

Look at the abelian case (class field theory):

$$G_{\mathbb{Q}_p}^{\mathrm{ab}} = \widehat{\mathbb{Q}_p^*}$$

We have the exact sequence

$$\mathbb{Z}_p^* \hookrightarrow \mathbb{Q}_p^* \twoheadrightarrow \mathbb{Z},$$

where the last arrow is the valuation v.

The Weil group W_p has a topology as follows: The kernel of $G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}$ is the inertia group I. Put the old topology on I and the discrete topology on $\mathbb{Z}$ (I is open).

One usually does not define a Weil-Deligne group, but only its representations. As a reference one can consult Tate's article in the Corvallis proceedings.

A *representation of the Weil-Deligne group* is

- a continuous representation $W_p \to \mathrm{GL}(V)$ for V a vector space with the discrete topology, typically over $\overline{\mathbb{Q}}$, and
- a nilpotent endomorphism N of V such that $wNw^{-1} = p^{d(w)}N$ for all w in W_p.

Last time we had a statement that under certain conditions there is a bijection of continuous l-adic representations and Weil-Deligne representations.

Now turn to the Bourbaki text from 2000, appendix B. One should look at some examples of Weil-Deligne representations..

Recall the exact sequence defining the inertia group. The wild inertia group is normal inside the inertia group and the quotient is called the tame inertia, which is the product of $\mathbb{Z}_l(1)$ with l-running through all primes not equal p. The tame group is also the projective limit of $\mathbb{F}_{p^n}^*$ running over all n.

("N =logarithm of monodromy") If you take $N : V \to V(1)$, then N is independent of choices.

Now finally the example I wanted to give. What is nice? There are a few things that are nice. If you work with modular forms, you automatically get representations to $\overline{\mathbb{Q}}$-vector spaces. On the Galois side, we always have to choose an l and the representation really depends on it. But if we work with the local Galois group we don't need that l, and that is good! But it is a non-trivial statement (if we don't believe local Langlands).

Question from audience: Is there an easier proof for this than local Langlands for modular forms? Answer: Yes, definitely. But in general not so easy.

The case of f (an eigenform) in weight 2 (then we have more geometry, we can work with abelian varieties). So let f be an eigenform for $\Gamma_1(N)$, character ϵ and weight 2. I try to describe the Weil-Deligne representation associated at the prime p by geometry. That has nothing l-adic in it!

We work with $J_1(N)_{\mathbb{Q}} = \mathrm{Pic}^0_{X_1(N)_{\mathbb{Q}}/\mathbb{Q}}$. Let $\mathbb{T}$ be the subring of $\mathrm{End}(J_1(N)_{\mathbb{Q}})$ generated by the Hecke operators T_n and the diamond operators. It is easier to stay at a finite level, but we could also do it in the direct limit.

Let K be $\mathbb{Q}(a_n(f))$, the coefficient field. Then f gives a morphism $\mathbb{T} \to \mathcal{O}_K$ sending t to $(t^*f)/f$.

Let $A_{\mathbb{Q}} := \mathcal{O}_K \otimes_{\mathbb{T}} J_1(N)_{\mathbb{Q}}$. If $\mathcal{O}_K = \mathbb{T}/I$, then $A_{\mathbb{Q}} = J_1(N)_{\mathbb{Q}}/IJ_1(N)_{\mathbb{Q}}$ and $A_{\mathbb{Q}}$ has multiplications by $\mathcal{O}_K$ (because we made it like that). Moreover, $\dim A_{\mathbb{Q}} = \dim_{\mathbb{Q}}(K)$.

Let $\mathbb{Q}_p \to F \subset \overline{\mathbb{Q}_p}$ be a finite Galois extension such that A_F has a semi-stable model A over O_F. A semi-stable model is a model of which the fibre at p is an extension of an abelian variety by a torus. Such a semi-stable model is unique.

The uniqueness of A implies that for all $\sigma \in G(F|\mathbb{Q}_p)$ we have the cartesian diagrams

$$\begin{array}{ccc} A & \xrightarrow{[\sigma]} & A \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}(\mathcal{O}_F) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\mathcal{O}_F) \end{array}$$

and with $O_F \twoheadrightarrow k_F$ (residue field)

$$\begin{array}{ccc} A_{k_F} & \xrightarrow{[\sigma]} & A_{k_F} \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}(k_F) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(k_F). \end{array}$$

<u>Idea</u>: Use $d : W_p \to \mathbb{Z}$ to undo the action on k_F.

We get: $W_p \to \left(\mathbb{Q} \otimes \operatorname{End}_{\mathcal{O}_K}(A_{k_F/k_F})\right)^*$. We send a $\sigma \in W_p$ to $[\sigma]^{-1}\operatorname{Frob}_{\text{abs}}^{d(\sigma)}$. (Frobenius commutes with everybody... It's an endomorphism of the identity functor.) You can do better by replacing $\mathbb{Q}$ by $\mathbb{Z}[1/p]$. This already is a piece of the Weil-Deligne representation.

Let us now describe the Weil-Deligne representation: We want the action on some vector space. Let's distinguish two cases. First that of good reduction. Then A_{k_F} is an abelian variety over k_F with the given multiplications by $\mathcal{O}_K$.

The ring $\mathbb{Q} \otimes \operatorname{End}_{\mathcal{O}_K}(A_{k_F/k_F})$ is a semi-simple K-algebra of dimension less or equal 4, so a quaternion algebra or a quadratic extension of K (it cannot be K). Let's remark that Frobenius commutes with $\mathcal{O}_K$.

Put

$$W_p \to \overline{\mathbb{Q}} \otimes_{\mathcal{O}_K} \operatorname{End}_{\mathcal{O}_K}(A_{k_F/k_F}) \subset \operatorname{GL}_2(\overline{\mathbb{Q}})$$

This is our Weil-Deligne representation if we embed it reasonably, that is to say that the embedding into $\operatorname{GL}_2(\overline{\mathbb{Q}})$ comes from embeddings of $\overline{\mathbb{Q}}$-algebras. Because of good reduction, N is zero, i.e. the monodromy is trivial. The l-adic representation becomes unramified over F (Serre-Tate).

We now do the case of bad reduction. Then A_{k_F} is a torus and $\mathbb{Q} \otimes \operatorname{Hom}(\mathbb{G}_{m,\overline{\mathbb{F}_p}}, A_{\overline{\mathbb{F}_p}})$ is a 1-dimensional K-vector space. So we have a character $\chi : W_p \to K^*$.

This determines a unique 2-dimensional Weil-Deligne representation over K with $N \neq 0$ and the given character on the kernel of N.

(The character determines the action on a line. How to make the N? Take $K \oplus K$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. On the first K take above character χ, on the other another character χ'. Then χ' is determined by the commutativity relation.)

Next time I'll give a kind of classification of 2-dimensional Weil-Deligne representations at primes $p > 2$. (For $p = 2$, the difficult case is when $N = 0$. There are cases when wild inertia does not act through characters, which can only happen if $p = 2$.)

2 Wednesday, 16 June 2004

Theo described earlier:

$$\lim_{\substack{\to \\ K}} H^1(\operatorname{Sh}_K(\operatorname{GL}_2(\mathbb{Q}), \mathbb{H}^{\pm})_{\overline{\mathbb{Q}},\text{et}} \mathcal{F}_{k,l}),$$

which has a $\operatorname{GL}_2(\mathbb{A}_f) \times G_{\mathbb{Q}}$-action and is equal to

$$\bigoplus_{f \text{ newform of weight k}} \pi_f \otimes (\rho_f^l)^{\vee}$$

with π_f a smooth irreducible representation of $GL_2(\mathbb{A}_f)$ on a $\overline{\mathbb{Q}_l}$-vector space. But it has a natural $\overline{\mathbb{Q}}$-structure.

We have $\pi_f = \bigotimes'_p \pi_{f,p}$, where $\pi_{f,p}$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $\overline{\mathbb{Q}}$-vector space. For all p we have $\rho^l_{f,p} := \rho^l_f|_{G_{\mathbb{Q}_p}}$ a continuous representation on a $\overline{\mathbb{Q}_l}$-vector space of dimension 2. Of course, ρ^l_f has no $\overline{\mathbb{Q}}$-model. However, locally it has one in terms of Weil-Deligne representations (cf. Gabor's talk in this seminar).

Recall the exact sequence

$$I_p \hookrightarrow G_{\mathbb{Q}_p} \twoheadrightarrow G_{\overline{\mathbb{F}_p}} = \widehat{\mathbb{Z}}.$$

The *Weil group* is defined as the preimage in $G_{\mathbb{Q}_p}$ of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$. The resulting map $d : W_p \to \mathbb{Z}$ is called the *degree map*. The statement now is the following:

> For all p and all l there exists a 2-dimensional $\overline{\mathbb{Q}}$-vector space V with a continuous action by W_p (we consider V with the discrete topology) and a nilpotent
>
> $$N : V \to V$$
>
> such that
>
> $$wNw^{-1} = p^{d(w)}N$$
>
> and:
>
> If $l \neq p$, then (V, N) gives us the $\rho^l_{f,p}$ as follows (depending on two choices: that of a Frobenius element Φ in W_p and that of a topological generator of $\lim\limits_{\leftarrow} \mu_{l^n}(\overline{\mathbb{Q}_p})$):
>
> $$\rho^l_{f,p}(\Phi^n\sigma) = \rho_{f,p}(\Phi^n\sigma)\exp(t_l(\sigma)N)$$
>
> for all $n \in \mathbb{Z}$ and all $\sigma \in I_p$.

As a reference one can consult Tate's Corvallis paper. If $l = p$, then we also have such a concept, using Fontaine functors.

Theorem 2 (Eichler-Shimura-Langlands, Deligne, Carayol) *For all p we have that $\rho^{F\text{-s.s.}}_{f,p}$ corresponds to $\pi_{f,p}$ via a suitably normalised local Langlands correspondence.*

F-semi-simplification means the semi-simplification as a W_p-representation (N acts automatically on it).

<u>Today:</u> I try to describe a little the $\rho_{f,p}$ en $\pi_{f,p}$.

One has the following *"Classification"* of the 2-dimensional F-s.s. WD-representations of $\mathbb{Q}_p$ over $\overline{\mathbb{Q}}$.

Let p be a prime number and (V, N) a 2-dimensional WD representation with V a $\overline{\mathbb{Q}}$-vector space. Suppose V is semi-simple as a W_p representation. Then (V, N) is in exactly one of the following:

(i) Decomposable: $N = 0$, $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

(ii) Indecomposable, but reducible as W_p-representation: $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (choose basis so that this holds) and $w \mapsto \begin{pmatrix} \alpha(w)p^{d(w)} & 0 \\ 0 & \alpha(w) \end{pmatrix}$

(iii) Irreducible, but inertia acts reducibly: $N = 0$, $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} \alpha$ with $\mathbb{Q}_p \hookrightarrow K$ a quadratic extension and $\alpha \neq \alpha^\sigma$ a character $\alpha : W_K \to \overline{\mathbb{Q}}^\times$.

(iv) I_p acts irreducibly: $N = 0, p = 2$. This is called *extra-ordinary*, and up to twist there exists only a finite number of them. Look at Weil: Exercises dyadiques.

Let's say something about the "proof".

For $N \neq 0$ it is simple, because the kernel of N is stable under the group. So suppose $N = 0$. Now we only have a Weil representation, the Deligne aspect has vanished.

There is the fact that every irreducible representation over $\overline{\mathbb{Q}}$ of a finite p-group has dimension a power of p. Thus is 2 a power of p, hence $p = 2$ as claimed. So assume $p \neq 2$. Suppose now we are in case (iii). Consider $\mathbb{P}(V)$ the set of lines in the vector space: it has trivial I_p-action or not. These two cases give the result, which is not difficult. In general the problems come from (iv).

Let's make a remark: What about the non-F-semi-simple ones? Their F-semi-simplifications are all in case (i).

For a fixed modular form f the Weil-Deligne representation $\rho_{f,p}$ is unramified for almost all p, actually for all $p \nmid \mathrm{level}(f) = \mathrm{conductor}(\pi_f)$.

The semi-simplification of $\rho_{f,p}$ corresponds to $\pi_{f,p}$. The left one is given by the two eigenvalues of Φ, and the right one? We want to describe this in the following case: $\rho_{f,p}^{F\text{-s.s.}} = \alpha \oplus \beta$.

For

$$\mu_1, \mu_2 : \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}^\times$$

we have $v : \mathbb{Q}_p^\times \to \mathbb{Z}$. Give μ_1, μ_2 by giving an image for the preimage of $1 \in \mathbb{Z}$. Make

$$V(\mu_1, \mu_2) := \{f : \mathrm{GL}_2(\mathbb{Q}_p) \to \overline{\mathbb{Q}} | f \text{continuous}, f(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}.g) = \mu_1(x)\mu_2(z)f(g)\}.$$

On it we have a $GL_2(\mathbb{Q}_p)$-action: $(g.f)h = f(h.g)$.

Another description. $V(\mu_1,\mu_2) = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mu_1,\mu_2)$ for the induction in a suitable category of representations. Here

$$\text{Borel} \twoheadrightarrow T = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}^\times$$

with $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto (x,z) \mapsto \mu_1(x)\mu_2(z)$.

Yet another description is $H^0_{\text{smooth}}(B\backslash G, V)$.

Why do we call this an induction?

$$\mathrm{Hom}_G(W, \mathrm{Ind}_B^G V) = \mathrm{Hom}_B(W,V)$$

Attention. Do not permute them (in case of finite groups in Frobenius reciprocity one may).

A remark: This is "naive" induction. B is not unimodular. People also use "unitary" induction (e.g. Langlands).

Theorem 3 *If $\mu_1\mu_2^{-1} \notin \{1, |.|_p^2\}$, then $V(\mu_1,\mu_2)$ is irreducible.*

Notation: $\pi(\mu_1,\mu_2)$ is called a principal series representation.

One has: $\pi(\mu_1,\mu_2) \cong \pi(\mu_1',\mu_2')$ if and only if $(\mu_1,\mu_2) = (\mu_1',\mu_2')$ or $(\mu_1',\mu_2') = (\mu_2|.|_p, \mu_1|.|_p^{-1})$.

Theorem 4 (Langlands) *If $\pi_{f,p} \cong \pi(\alpha,\beta)$, then $\rho_{f,p}^{F\text{-s.s.}} \cong \alpha|.|_p \oplus \beta$. (Local class field theory. $W_p \cong \mathbb{Q}_p^\times$)*

As a reference one can consult the Barcelona text by J.-B. Nortier.