The Tate curve

By Theo van den Bogaart. Notes of my talk held at the local Leiden seminar on algebraic geometry in November 2002. Ideas and structure by Bas Edixhoven.

Rough description

We are going to describe a particular scheme $\mathbf{Tate}(q)$ over $\mathbb{Z}[\![q]\!]$, called the *Tate curve*.

The motivation for constructing this curve comes from the analytic theory of elliptic curves. Here an elementary result says that every complex elliptic curve is isomorphic to $\mathbb{C}/\Lambda$ for a lattice Λ, which we may choose as $\Lambda = \mathbb{Z}+\tau\mathbb{Z}$ for some number τ in the complex upper half plane. Conversely, any such Λ determines an elliptic curve. Since the group law in $\mathbb{C}/\Lambda$ is very transparent (e.g., torsion points are immediately visible) this is a very useful characterization. The exponential map $e\colon z \mapsto e^{2\pi i z}$ induces an isomorphism

$$\mathbb{C}/\Lambda \xrightarrow{\sim} \mathbb{C}^*/q^{\mathbb{Z}},$$

where $q^{\mathbb{Z}}$ denotes the free multiplicative group generated by $q = e(\tau)$. Reformulating the above, we see that a complex elliptic curve is isomorphic to $\mathbb{C}^*/q^{\mathbb{Z}}$ for some non-zero complex number q with $|q| < 1$ and every such q determines an elliptic curve. Thus we get a family of elliptic curves

$$T \longrightarrow D^*$$

over the punctured unit disc: the fibre above $q \in D^*$ is $\mathbb{C}^*/q^{\mathbb{Z}}$. We can construct an embedding of T into projective space — in each fibre this gives a Weierstrass equation, whose coefficients depend holomorphically on the parameter q. Using the well-known formulas, we can thus expand the associated values j, Δ, ... as Laurent series in q. This gives for example the famous expansion

$$j(q) = q^{-1} + 744 + 196884q + \cdots .$$

We can conveniently write the Weierstrass equation obtained as

$$y^2 + xy = x^3 + a_4(q)x + a_6(q). \tag{$*$}$$

Turning things around, from now on we regard the coefficients $a_4(q)$ and $a_6(q)$ just as formal power series in q. Tate observed that $(*)$ then defines a projective curve, not just in $\mathbb{P}^2_{\mathbb{C}[\![q]\!]}$, but even in $\mathbb{P}^2_{\mathbb{Z}[\![q]\!]}$. It is clear that this *is* an elliptic curve over the ring of Laurent series $\mathbb{Z}(\!(q)\!) := \mathbb{Z}[\![q]\!](q^{-1})$. Here are some nice consequences:

i) Calculating j, Δ, ... from $(*)$, we understand why for example the q-expansion of the j-invariant has integer coefficients.

ii) Tate's construction makes it clear that we can perform the same trick over some non-archimedian field K. The requirement that $|q| < 1$ will suffice to make all the power series involved converge. A theorem by Tate asserts that if K is a p-adic algebraically closed field of characteristic 0, then there is a bijective correspondence between numbers $q \in K$ such that $0 < |q| < 1$ and isomorphism classes of elliptic curves with j-invariant $|j| > 1$, given by $q \mapsto K^*/q^{\mathbb{Z}}$ (compare this with the complex case).

iii) The Tate curve can be used to describe the boundary of the moduli space of elliptic curves. If $X(1)_{\mathbb{Z}[\frac{1}{n}]}$ is the compactified moduli space, then the Tate curve determines a classifying morphism

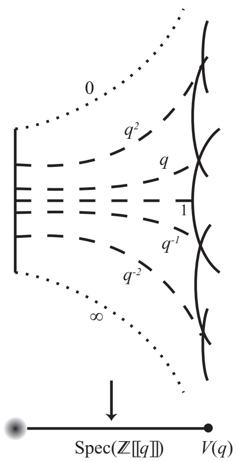
$$\mathbb{Z}[\tfrac{1}{n}][\![q]\!] \longrightarrow X(1)_{\mathbb{Z}[\frac{1}{n}]}$$

which (roughly) identifies $\mathbb{Z}[\frac{1}{n}][\![q]\!]$ with the completion of the boundary.

Geometric construction

In this section we give a geometric construction of the Tate curve. The advantage of this aproach is that a lot of properties become easier to prove — we will list some of them in the last section.

First some heuristic remarks motivating the construction which follows. We would like to find some sort of algebraic analogue for the construction of the quotient $\mathbb{C}^*/q^{\mathbb{Z}}$ for holomorphic spaces. First we forget about the quotient and just look at $\mathbb{G}_{m,\mathbb{C}}^{\mathrm{hol}}$. We want a scheme T (over $\mathbb{C}[\![q]\!]$ say, to fix the idea; but its construction works over $\mathbb{Z}[\![q]\!]$ as well and we soon generalize to this situation) whose formal completion (at the closed fibre) is naturally associated to $\mathbb{G}_{m,\mathbb{C}}^{\mathrm{hol}}$. Motivated by result of Raynaud (cf. the discussion in the introduction to [2]), one finds a construction of T over $\mathbb{Z}[\![q]\!]$ by starting with the projective line and blow it up an infinite number of times in 0 and ∞ in the fibre $q = 0$. We will give a precise definition later, but the picture is this:



This picture suggests there is a natural $\mathbb{Z}$-action on T given by multiplication with q. Unfortunately, we cannot just take the quotient, for this gives problems outside the special fibre $V(q)$. But restricting to $V(q)$ everything works fine. More generally, we restrict to infinitesimal neighbourhoods T_n of the special fibre. Thus the $T_n/q^{\mathbb{Z}}$ give a formal scheme, which is algebraizable by the Tate curve $\mathbf{Tate}(q)$. Viewed differently: we can embed $T_n/q^{\mathbb{Z}}$ in projective space, the image of this embedding being given by a Weierstrass equation $(*)$. As n gets larger, the coefficients $a_4(q)$ and $a_6(q)$ of this equation acquire higher and higher powers of q. In the limit this gives the power series searched for. We work out the above programme in three steps.

Step 1: Blowing up. Put $X_0 = \mathbb{P}^1_{\mathbb{Z}\llbracket q\rrbracket}$, the projective line over $\mathbb{Z}\llbracket q\rrbracket$; in the standard way we fix two (non-closed) points 0 and ∞ on it. In general, for some $i \geq 0$ suppose there is given a $\mathbb{Z}\llbracket q\rrbracket$-scheme X_i together with two points on it, denoted also (by abuse of notation) 0 and ∞. Define X_{i+1} as the blow-up of X_i along the subscheme obtained by taking the closure of 0 and ∞ and intersecting it with $V(q)$ (and giving this its reduced induced subscheme structure). It comes with a morphism $f_i\colon X_{i+1} \to X_i$. Define two points on X_{i+1}, denoted by abuse of notation also by 0 and ∞, as the generic points of $f_i^{-1}(\overline{0} \setminus V(q))$ and $f_i^{-1}(\overline{\infty} \setminus V(q))$ respectively. With X_0 as above, we have inductively defined a collection of triples $(X_i, 0, \infty)_{i\geq 0}$ and morphisms $f_i\colon X_{i+1} \to X_i$. In order to take the limit, we reverse the arrows: Let V_i be X_i with 0 and ∞ removed. Thus we get a system $\{V_i\}$ of schemes together with open embeddings $V_i \hookrightarrow V_{i+1}$ (note however that the inverse mapping f_i no longer exists). Define $T = \varinjlim V_i$; this is well-defined: as a topological space we take the union of all the V_i and the structure sheaf we get from glueing the $\mathcal{O}_{V_i}$. Note that T is a scheme over $\mathbb{Z}\llbracket q\rrbracket$ which is not quasi-compact.

To get a better understanding of what is going on, let us elaborate on the blow-ups at 0 (the blow-ups at ∞ are handled in symmetric fashion). For each i, we can look at an affine open neighbourhood U_i around 0 in X_i. Even better: we can take U_i as the affine line over $\mathbb{Z}\llbracket q\rrbracket$; indeed, for $i = 0$ this is clear and for $i > 0$ this is explained inductively by the following diagram:

$$\begin{tikzcd}
X_{i-1} & X_i \arrow[l, "f_{i-1}"'] & \\
U_{i-1} \arrow[u, hook] \arrow[d, "\wr", "\text{(i.h.)}"'] & f_{i-1}^{-1}(U_{i-1}) \arrow[u, hook] \arrow[l] \arrow[dd, "\wr"] & U_i \arrow[l, hook'] \arrow[d, "\wr"] \\
\operatorname{Spec} \mathbb{Z}\llbracket q\rrbracket[x] & & \operatorname{Spec} \mathbb{Z}\llbracket q\rrbracket[x][X]/(qX-x) \arrow[d, "\wr"] \\
& \operatorname{Proj}(\mathbb{Z}\llbracket q\rrbracket[x])[X,Y]/(qX-xY) & \operatorname{Spec} \mathbb{Z}\llbracket q\rrbracket[X] \arrow[uull, dashed, bend left, "f_{i-1}|_{U_i}"]
\end{tikzcd}$$

Clearly, $f_{i-1}|_{U_i}$ is induced by the mapping $x \mapsto qX$ between the corresponding rings. But there are two other obvious maps between U_{i-1} and U_i:

$$\begin{aligned}
q\colon U_{i-1} \to U_i \qquad &\text{given by} \qquad X \mapsto x \\
q^{-1}\colon U_i \to U_{i-1} \qquad &\text{given by} \qquad x \mapsto X
\end{aligned}$$

which are inverse to each other. For future convenience, we define $W_i = U_i \setminus 0$.

We view W_i as an open subscheme of T; it is isomorphic (as a scheme) to the group scheme $\mathbb{G}_{m,\mathbb{Z}[\![q]\!]}$.

In an analogous way, we can look at the blow-ups at ∞ and get compatible maps q, q^{-1}. These maps extend to the limit T. We can also extend the definition of W_i to $i < 0$. Notice that in the fibres where q is invertible, T is just the multiplicative group scheme. Restricting q and q^{-1} to these fibres, we get multiplication maps. We will therefore call the map q *multiplication by* q.

Thus we have defined a non-compact scheme T over $\mathbb{Z}[\![q]\!]$ equiped with a natural $\mathbb{Z}$-action given by multiplication by q.

Step 2: Taking quotients. For $n \geq 1$, define T_n as the fibre above the n-th infinitesimal neigbourhood of $V(q)$; e.g.,

$$T_n = T \times_{\operatorname{Spec}\mathbb{Z}[\![q]\!]} \operatorname{Spec}\mathbb{Z}[\![q]\!]/(q^n).$$

We can pull-back the multiplication maps $q^{\cdot}\colon T \to T$ to T_n. Now the quotient $T_n/q^{\mathbb{Z}}$ exists; to see this, just note that for any k with $|k| \geq 2$ the map q^k just identifies de *disjoint* parts V_k and V_{i+k}, thus $T_n/q^{2\mathbb{Z}}$ results from a glueing operation. All that remains is to take the finite quotient by $\mathbb{Z}/2\mathbb{Z}$.

The data $(T_n/q^{\mathbb{Z}})$ defines a formal scheme $\mathfrak{Tate}(q)$ over $\operatorname{Spf}\mathbb{Z}[\![q]\!]$. One consequence of the next step is that each $\mathcal{T}_n/q^{\mathbb{Z}}$ has an ample line bundle, forming a compatible system. By Grothendieck's algebraizability theorem [EGA III.5.4.5] we see that there exists a unique scheme $\mathbf{Tate}(q)$ over $\mathbb{Z}[\![q]\!]$ such that $\mathfrak{Tate}(q)$ is the formal completion of $\mathbf{Tate}(q)$ along the fibre above $V(q)$. This scheme $\mathbf{Tate}(q)$ is the Tate curve.

Step 3: Ample line bundle. On T define the line bundle $\mathcal{L} = \mathcal{O}_T(D)$ given by the divisor

$$D = \sum_{m\in\mathbb{Z}} \frac{m^2 - m}{2} D_m,$$

where D_m is the irreducible component of the fibre of T above $V(q)$ to which q^m specializes (this is just the closure of $W_m \cap V(q)$ of the above discussion); note that each D_m is isomorphic to the projective line over $\mathbb{Z}$. Pull back $\mathcal{L}$ to form line bundles $\mathcal{L}_n$ on T_n. We will look for sections of $\mathcal{L}_n$ that are invariant under the $\mathbb{Z}$-action.

(*Heuristic remark:* How to arrive at the precise form of D above? This follows from a clever guess by looking at the way things work for the classical construction of the Weierstrass equation for an elliptic curve. Indeed, suppose that instead of $\mathbb{Z}[\![q]\!]$, we work over $\mathbb{Q}[\![q]\!]$. We can then talk locally (on a quasi-compact piece) about the degree of D. Write $D = \sum a_m D_m$, where we want to determine the a_m. Now a comparison with the classical theory makes it natural to demand that $\deg(D|_{D_m}) = 1$. Working this out using intersection products gives

$$\begin{aligned}1 = (D.D_m) = a_{m-1}(D_{m-1}.D_m) + a_m(D_m.D_m) + a_{m+1}(D_{m+1}.D_m)\\ = a_{m-1} + a_m(D_m.D_m) + a_{m+1}.\end{aligned}$$

We determine the self-intersection number by the following trick. The constant function q on T has $\operatorname{div}(q) = \sum D_m$ and, being a principal divisor, its degree is zero; so

$$0 = (\mathrm{div}(q).D_m) = (D_{m-1}.D_m) + (D_m.D_m) + (D_{m+1}.D_m) = 2 + (D_m.D_m).$$

This results in the recurrence relation

$$a_{m-1} - 2a_m + a_{m+1} = 1$$

which indeed has as a solution $a_m = \frac{m^2-m}{2}$.)

The multiplication map q transforms $\mathcal{L}$ as follows:

$$\begin{aligned} q^*\mathcal{L} = q^* \left(\mathcal{O}_T(\sum \frac{m^2-m}{2} D_m) \right) \\ = \mathcal{O}_T(\sum \frac{m^2-m}{2} D_{m-1}) = \mathcal{O}_T(\sum \frac{m^2+m}{2} D_m). \end{aligned}$$

With the notation of step 1, we have a natural function $x \in \mathcal{O}_{W_0}$ (recall that $W_0 \simeq \mathrm{Spec}\,\mathbb{Z}[\![q]\!][x, x^{-1}]$), giving a rational function on the whole of T. Noticing that

$$\mathrm{div}(x) = \sum m D_m = \sum \frac{m^2+m}{2} D_m - \sum \frac{m^2-m}{2} D_m$$

we see that multiplication by $-x$ gives an isomorphism

$$\mathcal{L} \longrightarrow q^*\mathcal{L}$$

(the minus sign is there to get a correctly normalized theta function later on). Therefore, invariant sections are given by functions θ such that

$$\theta(q^{-1}x) = -x\theta(x).$$

We apply the old trick for constructing such sections: just let q act on an arbitrary section an infinite number of times and (roughly said) add up all these contributions. To be explicit: since D is effective, we have $1 \in \Gamma(T, \mathcal{L})$; this section gives

$$\cdots \mapsto qx^{-2} \mapsto -x^{-1} \mapsto 1 \mapsto -qx \mapsto q^3x^2 \mapsto \cdots \mapsto (-1)^m q^{\frac{m^2+m}{2}} x^m \mapsto \cdots$$

hence the invariant expression

$$\theta(x) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{m^2+m}{2}} x^m.$$

This is properly defined as we regard it as a section of $\mathcal{L}_n$ over T_n.

From now on, one has to do a lot of calculations. Let us just describe how to look them up in the literature. To start with, the function $\theta(x)$ has product expansion

$$\prod_{n \geq 0} (1 - q^n x^{-1}) \prod_{n > 0} (1 - q^n x)$$

up to an invertible element of $\mathbb{Z}[\![q]\!]$ — see Siegel's book [5], esp. page 38; he proves it in the complex analytic setting[1]. This leads to Roquette's definition of

[1]In Siegel's notation: $q = e^{2\pi i z}$, $x = e^{2\pi i w}$ and his θ_1 differs from our θ by a multiplicative constant and a factor $\sqrt{x}$.

the theta-function. On pages 24–34 of his book [4] he gives the formulas for an embedding of T_n in $\mathbb{P}^2$ and eventually he also calculates the explicit expansion of the j-invariant[2]. See also the calculations in chapter II.5 in the book of Robert [3].

Properties

Let us collect some properties of the Tate curve which easily follow from the above construction:

Group structure The smooth part of $\mathbf{Tate}(q)$ is a group scheme. Its singular locus lives in the fibre above $q = 0$; over $\mathbb{Z}((q))$, the Tate curve is an elliptic curve.

Torsion points The n-torsion points of the smooth part of the Tate curve are immediately vissible, namely $\xi_n^a(q^{1/n})^b$ with $0 \leq a, b < n$ (and ξ_n a primitive n-th root of unity).

Invariant differential There is a global differential form $\frac{dx}{x}$ on $\mathbf{Tate}(q)$.

For more details, see [1].

References

[1] P. Deligne, M. Rapoport: Les Schémas de Modules de Courbes Elliptiques; *Modular Functions of One Variable II*, Springer LNM **349** (1973), pp. 143–316.

[2] David Mumford: Degenerating Curves; *Comp. Math.* **24** (1972), pp. 129–174.

[3] Alain Robert: *Elliptic Curves*; Springer LNM **326** (1973).

[4] Peter Roquette: *Analytic Theory of Elliptic Functions over Local Fields*; Vandenhoeck & Ruprecht (1970).

[5] Carl Ludwig Siegel: *Advanced Analytic Number Theory*; Tata Institute of Fundamental Research (1980).

[2] Roquette constructs the so-called Weierstrass $\wp$-function. To link this up with what we are doing here: we derived $\theta_1 := \theta$ as a global section of $\mathcal{L}_n$. Now θ_1^2 and $\theta_1^2\wp$ are sections of $\mathcal{L}_n^{\otimes 2}$. In addition we get $\theta_1^3\wp'$ as a global section of $\mathcal{L}_n^{\otimes 3}$.