Tentamen Algebra 3, 19 juni 2014, 13:00–17:00

Bas Edixhoven

During this exam electronic equipment is not allowed. Allowed are: books, syllabi and notes. An indicative weighting of the exercises is given at the bottom of page 2. There are 4 exercises. The exam wil be graded on June 21. Success!

Opgave 1. Let $f = X^4 - 9$ in $\mathbb{Q}[X]$.

(a) Determine the set N of zeros of f in $\mathbb{C}$.

(b) Determine the splitting field $\Omega_{\mathbb{Q}}^f \subset \mathbb{C}$: give a basis over $\mathbb{Q}$.

(c) Determine $\mathrm{Gal}(\Omega_{\mathbb{Q}}^f/\mathbb{Q})$, and give the corresponding permutations of N.

(d) Give a primitive element α of $\Omega_{\mathbb{Q}}^f$ over $\mathbb{Q}$, and the minimal polynomial $f_{\mathbb{Q}}^{\alpha}$.

(e) Write α^{-1} in the basis of powers of α.

Opgave 2. Let $\mathbb{F} := \mathbb{F}_{64}$. Note that $64 = 2^6$.

(a) How many subfields does $\mathbb{F}$ have, how many elements does each of them have, and how many of those generate the subfield?

(b) Determine the number of irreducible polynomials of degree 6 in $\mathbb{F}_2[X]$.

(c) Show that $\mathbb{F}$ is a splitting field of the polynomial Φ_9 in $\mathbb{F}_2[X]$.

(d) Let $\zeta \in \mathbb{F}$ be a zero of Φ_9. Give all zeros of Φ_9 in $\mathbb{F}$, expressed in ζ.

(e) Show that Φ_9 is irreducible in $\mathbb{F}_2[X]$.

Opgave 3. Let $\zeta = e^{2\pi i/7}$ in $\mathbb{C}$. For subsets T of $\mathbb{F}_7^*$ we define

$$z_T := \sum_{a \in T} \zeta^a.$$

(a) Give the list of subfields of $\mathbb{Q}(\zeta)$, and for each subfield a generator.

(b) Give a subset T of $\mathbb{F}_7^*$ with $\#T = 3$ for which z_T is constructible with straight-edge and compass from $\{0, 1\}$.

(c) Determine all subsets T of $\mathbb{F}_7^*$ for which z_T is constructible with straight-edge and compass from $\{0, 1\}$.

Opgave 4.

(a) Do there exist a field K and an *irreducible* separable polynomial f over K of degree 7 with $\mathrm{Gal}(\Omega_K^f/K)$ isomorphic to the symmetric group S_6?

(b) Determine the Galois group $\mathrm{Gal}(\Omega_{\mathbb{Q}}^f/\mathbb{Q})$ of $f = X^5 - 6$ as subgroup of S_5 by giving its order and generators for it.

(c) Show that for every $n \in \mathbb{Z}_{>0}$ and every transitive subgroup G of S_n there exist a field K and an *irreducible* separable polynomial f over K of degree n, such that $\mathrm{Gal}(\Omega_K^f/K)$ is isomorphic to G. Hint: first make a Galois extension $K \subset L$ with group S_n.

(d) Do there exist a field K and an *irreducible* separable polynomial f over K of degree 6 with $\mathrm{Gal}(f)$ isomorphic to the symmetric group S_5?

Normering (indicatief): 100 = 10 (gratis) + 25 (5x5) + 20 (5x4) + 21 (3x7) + 24 (4x6)